Algebraic aspects of topological quantum field theories

Arik Wilbert

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Betreuer: Prof. Dr. Catharina Stroppel

Mathematisches Institut

Mathematisch-Naturwissenschaftliche Fakultät der

Rheinischen Friedrich-Wilhelms-Universität Bonn

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Introduction

In this thesis it is examined how certain algebraic structures arise in the study of topological quantum field theory. The notion of a *topological quantum field theory (TQFT)* was coined by Witten [Wit88] in 1988. The physically motivated idea is to provide a mathematical framework for studying quantum field theory which does not depend on the Riemannian metric of the underlying space-time manifold. In this sense the theory is purely topological. In his paper Witten already foreshadows the possibility of using TQFTs to construct meaningful manifold invariants. This was probably the starting point for mathematicians to become interested in TQFTs as well. Shortly after, Atiyah [Ati89] proposed a set of axioms which were supposed to lay a rigorous foundation for a mathematical treatment of TQFTs.

Based on the ideas presented in Atiyah's paper a *n-dimensional TQFT* can be thought of as a rule which assigns finite-dimensional vector spaces to closed oriented $(n-1)$-manifolds and linear maps to n-dimensional oriented cobordisms (up to diffeomorphism preserving the boundary) between two such $(n-1)$-manifolds. Using the modern language of category theory we will speak of a n-dimensional TQFT as a functor from the cobordism category **nCob** to the category $\mathbf{Vect}_k$ of finite-dimensional vector spaces which obeys certain additional properties.

According to a general rule of quantum mechanics, many-particle systems are described by the tensor product of the particles' state spaces. Thus it is natural to demand from a TQFT that the functor sends the disjoint union of $(n-1)$-manifolds (each one corresponding to a particle) to the tensor product of the assigned vector spaces (corresponding to the respective state spaces). We will see that a mathematician would describe this situation by saying that the category **nCob** is a *monoidal category* with respect to disjoint union and so is $\mathbf{Vect}_k$ with respect to the usual tensor product. Moreover, the functor preserves this structure and is therefore called a *monoidal functor*. Introducing these notions in detail and providing some important examples is actually the aim of the first section. In conclusion a n-dimensional TQFT is a monoidal functor $\mathbf{nCob} \to \mathbf{Vect}_k$.

Quinn [Qui95] has been one of the first people to realize the mathematical potential of describing TQFTs in terms of category theory. He also suggests to replace the cobordism category by some other category which might be of combinatorial or algebraic flavor. We will adopt this viewpoint and thus define a general TQFT as a monoidal functor $\mathcal{C} \to \mathbf{Vect}_k$ where the monoidal category $\mathcal{C}$ can be specified at our own discretion.

In this thesis we are interested in two particular choices of monoidal categories $\mathcal{C}$. For the largest part we will be concerned with the classical Atiyah-type TQFTs $\mathbf{nCob} \to \mathbf{Vect}_k$ described above. In order to understand these functors we will spend an entire section on oriented cobordisms and construct the category **nCob**. The connection to algebra arises in the special case $n = 2$. This is basically due to the fact that the category **2Cob** can be described explicitly by a set of generators and relations. In turn this is only possible because a complete classification of 2-manifolds/surfaces exists. Eventually the ultimate goal will be to use this result to describe 2d-TQFTs only in terms of algebra.

The main result in this context states that there is a bijection between (symmetric) monoidal functors $\mathbf{2Cob} \to \mathbf{Vect}_k$ and commutative Frobenius algebras. In other words given a 2d-TQFT there is a corresponding Frobenius algebra and vice versa. To prepare ourselves for the proof, this algebraic structure is studied extensively in Section 3. Frobenius algebras can be characterized as algebras that come with a certain linear form or equivalently with an associative, non-degenerate pairing. The aim is to see that these algebras can be equipped with a special structure of a coalgebra. This turns out to be the crucial property of Frobenius algebras regarding their connection with 2d-TQFTs. The construction of the comultiplication will be done using a graphical calculus following Kock [Koc04]. In fact, the excellent (but lengthy) book by Kock will be the main source for our discussion of classical TQFTs. After all these general considerations some explicit examples are studied in 4.2. In particular the way in which TQFTs produce manifold invariants will be outlined.

At the end of the thesis we present an outlook by replacing the category of cobordisms by a *k-linear abelian monoidal category* in the spirit of Quinn. The functors which are interesting in this context are *fiber functors*. They correspond to bialgebras in a way which is very similar to the correspondence between classical TQFTs and Frobenius algebras. The underlying theory of this result is the theory of Tannaka reconstruction. This goes a lot further than what is presented in this thesis and it is an interesting subject in its own right, see [JS91]. The discussion presented here is based on the lecture notes of a course on tensor categories given at MIT, cf. [EGNO]. A positive aspect about these notes is that they are goal-oriented and quickly come to the significant results without detours. However, the flip side is that most of the proofs are left out or posed as exercises. This motivated to work out this text and fill in some details. Hopefully, this thesis presents a short introduction to the basics of reconstruction of bialgebras which is readable for people without prior experience in this area.

Acknowledgements: I would like to thank Hanno Becker for providing me with some useful hints regarding the proof of Theorem 42. Last but not least, special thanks go to Prof. Dr. Catharina Stroppel for advising this thesis and suggesting to incorporate the basic ideas of reconstructionism in addition to classical TQFTs.

1 Topological quantum field theory and monoidal categories

We begin by defining abstractly the central object of study. Throughout this text $\mathbf{Vect}_k$ denotes the category of finite-dimensional vector spaces over some fixed field k.

Definition 1. Let $\mathcal{C}$ be a monoidal category. A *topological quantum field theory (TQFT)* is a monoidal functor $F : \mathcal{C} \to \mathbf{Vect}_k$.

This first section is devoted to explaining the contents of this definition and to introducing some closely related notions and results from category theory which will be important throughout this thesis. We will assume basic knowledge about categories, functors[1] and natural transformations as provided by [ML98]. The definitions given in this section can be found in any source on monoidal categories, e.g. [EGNO] or [Kas95].

The notion of a monoidal category generalizes the concept of the tensor product which we are familiar with from the category $\mathbf{Vect}_k$ (cf. Example 4). Precisely we have the following

Definition 2. A *monoidal category*[2] is a sextuple $(\mathcal{C}, \otimes, a, \mathbf{1}, l, r)$ where $\mathcal{C}$ is a category, $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ is a functor called the *tensor product*, a is a natural isomorphism

$$a_{X,Y,Z} : (X \otimes Y) \otimes Z \xrightarrow{\sim} X \otimes (Y \otimes Z) \qquad \forall X, Y, Z \in \mathcal{C}$$

called the *associativity constraint*, $\mathbf{1} \in \mathcal{C}$ is an object, l and r are natural isomorphisms

$$l_X : \mathbf{1} \otimes X \xrightarrow{\sim} X \qquad \forall X \in \mathcal{C}$$
$$r_X : X \otimes \mathbf{1} \xrightarrow{\sim} X \qquad \forall X \in \mathcal{C}$$

called the *unit constraints*. This data is subject to the following axioms:

1. **(Pentagon Axiom)** The diagram

$$\begin{array}{ccccc} & & ((W \otimes X) \otimes Y) \otimes Z & & \\ & \swarrow^{a_{W\otimes X,Y,Z}} & & \searrow^{a_{W,X,Y}\otimes id_Z} & \\ (W \otimes X) \otimes (Y \otimes Z) & & & & (W \otimes (X \otimes Y)) \otimes Z \\ \downarrow a_{W,X,Y\otimes Z} & & & & \downarrow a_{W,X\otimes Y,Z} \\ W \otimes (X \otimes (Y \otimes Z)) & & \xleftarrow{id_W \otimes a_{X,Y,Z}} & & W \otimes ((X \otimes Y) \otimes Z) \end{array}$$

is commutative.

[1] Unless stated otherwise all functors are supposed to be covariant.

[2] Some authors refer to monoidal categories as *tensor categories*.

2. **(Triangle Axiom)** The diagram

$$\begin{array}{ccc} (X\otimes \mathbf{1})\otimes Y & \xrightarrow{a_{X,\mathbf{1},Y}} & X\otimes(\mathbf{1}\otimes Y)\\ & {\scriptstyle r_X\otimes id_Y}\searrow \qquad \swarrow{\scriptstyle id_X\otimes l_Y} & \\ & X\otimes Y & \end{array}$$

is commutative.

These are called *coherence diagrams*.

Remark 3. The name "monoidal category" originates from the fact that this structure can be thought of as a categorification of a monoid. Recall that a monoid is simply a set together with an associative multiplication map and a neutral element. This concept can be lifted to the level of categories by replacing the set with a (small) category and elements of the set by objects. The multiplication map then corresponds to the tensor functor. Equalities are replaced by isomorphisms and thus the associativity of the mulitplication translates into the associativity constraint and the properties of the unit of a monoid simply become the unit constraints.

Example 4. As already mentioned the standard example of a monoidal category is $\mathbf{Vect}_k$. In this case the tensor functor assigns to a pair of k-vector spaces V, W their usual tensor product $V \otimes_k W$ over k and to a pair of maps $\varphi : V \to W$, $\psi : V' \to W'$ the tensor product map $\varphi \otimes \psi : V \otimes V' \to W \otimes W'$ given by $x \otimes y \mapsto \varphi(x) \otimes \psi(y)$. The associativity constraint is realized by the canonical isomorphism $(V \otimes W) \otimes X \xrightarrow{\sim} V \otimes (W \otimes X)$ defined by $(v \otimes w) \otimes x \mapsto v \otimes (w \otimes x)$. Moreover, the unit object is k, and the unit constraints $k \otimes V \xrightarrow{\sim} V$ and $V \otimes k \xrightarrow{\sim} V$ are given by $\lambda \otimes v \mapsto \lambda.v$ and $v \otimes \lambda \mapsto \lambda.v$ respectively. A straightforward calculation shows that these constraints satisfy the pentagon and triangle axiom.

Example 5. The following example will become important later (see 4.3). Let H be a finite-dimensional algebra over k. Consider $\mathbf{Rep}(H)$, the category of finite-dimensional[3] representations of H. Objects of this category are pairs (V,ϕ) where V is a finite-dimensional k-vector space and $\phi : H \to \mathrm{End}_k(V)$ is a unital algebra homomorphism. Equivalently, objects of $\mathbf{Rep}(H)$ can be thought of as H-modules where the action of H is given by $h.v := \phi(h)(v)$. A morphism $(V, \phi) \to (W, \psi)$ is a k-linear map $f : V \to W$ such that $f(\phi(h)(v)) = \psi(h)(f(v))$ for all $h \in H$ and $m \in V$, or alternatively, a homomorphism of H-modules. It is easy to check that this is indeed a category.

[3]The finiteness condition on both the dimension of the algebra and the representations can be dropped. But we will be interested in the finite-dimensional case exclusively.

If H is also a bialgebra with structure maps $\mu, \eta, \delta, \varepsilon$ (cf. Appendix A) we can construct a monoidal structure by defining $(V, \phi) \otimes (W, \psi)$ to be the pair consisting of the vector space $V \otimes W$ and the algebra homomorphism

$$H \xrightarrow{\delta} H \otimes H \xrightarrow{\phi \otimes \psi} \mathrm{End}_k(V) \otimes \mathrm{End}_k(W) \xrightarrow{c} \mathrm{End}_k(V \otimes W)$$

where δ denotes the comultiplication of H and the map c is the obvious one. The tensor product of two morphisms is simply the tensor product of the two k-linear maps.

We define an associativity constraint

$$a_{(V,\phi),(W,\psi),(X,\sigma)} : ((V, \phi) \otimes (W, \psi)) \otimes (X, \sigma) \xrightarrow{\sim} (V, \phi) \otimes ((W, \psi) \otimes (X, \sigma))$$

by the canonical isomorphism $a_{V,W,X} : (V \otimes W) \otimes X \xrightarrow{\sim} V \otimes (W \otimes X)$ of vector spaces (cf. Example 4). For this to be an isomorphism in $\mathbf{Rep}(H)$ it needs to be checked that $a_{V,W,X}$ is an isomorphism of H-modules. This is a straightforward calculation using the coassociativity of δ. Instead of carrying it out explicitly, let us discuss the unit object a bit more detailed.

The unit object is given by (k, ε) where $\varepsilon : H \to k$ is the counit of H (since there is a canonical isomorphism $k \cong \mathrm{End}_k(k)$ given by $\lambda \mapsto (1 \mapsto \lambda \cdot 1)$ we identify k with $\mathrm{End}_k(k)$). To define a unit constraint $(k, \varepsilon) \otimes (V, \phi) \xrightarrow{\sim} (V, \phi)$ in $\mathbf{Rep}(H)$ we use the isomorphism $l_V : k \otimes V \to V$ given by the action of k on V (cf. Example 4). It remains to check that

$$l_V(c \circ \varepsilon \otimes \phi \circ \delta(h)(1 \otimes v)) = \phi(h)(l_V(1 \otimes v)).$$

First we factor

$$\varepsilon \otimes \phi \circ \delta = id \otimes \phi \circ \underbrace{\varepsilon \otimes id \circ \delta}_{h \mapsto 1 \otimes h}$$

and use the counit axiom. Thus one gets

$$\varepsilon \otimes \phi \circ \delta(h) = 1 \otimes \phi(h)$$

where 1 is identified with the identitiy endomorphism of k. Applying c yields

$$c \circ \varepsilon \otimes \phi \circ \delta(h) = id_k \otimes \phi(h)$$

All in all we have

$$l_V(c \circ \varepsilon \otimes \phi \circ \delta(h)(1 \otimes v)) = l_V(id_k \otimes \phi(h)(1 \otimes v)) = \phi(h)(v) = \phi(h)(l_V(1 \otimes v)).$$

Analogously it can be shown that (k, ε) is also a right unit. Moreover, one can convince oneself that all this data actually satisfies the coherence diagrams.

Definition 6. A monoidal category is called *strict* if for all objects X, Y, Z in $\mathcal{C}$ one has equalities $(X \otimes Y) \otimes Z = X \otimes (Y \otimes Z)$ and $\mathbf{1} \otimes X = X = X \otimes \mathbf{1}$, and the associativity and unit constraints are the identity maps.

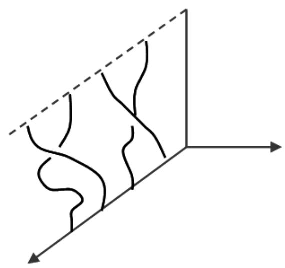
Until now all examples under consideration did not have the property of being strict (the constraint maps were isomorphisms but not identities). In the following, two examples of strict monoidal categories are discussed.

Example 7. The standard example of a strict monoidal category is the category of endofunctors of a given category. More precisely, let $\mathcal{C}$ be any category (not necessarily monoidal). Consider the category $\mathrm{End}(\mathcal{C})$ of all functors from $\mathcal{C}$ to itself (morphisms in this category are natural transformations). Then the tensor product in $\mathrm{End}(\mathcal{C})$ is simply the composition of functors. The associativity constraint is given by the identity natural transformation. Moreover, the unit object in $\mathrm{End}(\mathcal{C})$ is defined to be the identity functor and the respective constraints are the identity natural transformation again. Hence we have a strict monoidal category.

Example 8. Another important example of a strict monoidal category is the category **Braid** of braids. Before explaining this category we recall some basic facts about braids in general; see [KT08, 1.2.1].

A *geometric braid* on $n \geq 1$ strings is a set $b \subset \mathbb{R}^2 \times [0,1]$ formed by n disjoint topological intervals (a topological space homeomorphic to $[0,1]$) called the strings of b such that the projection $\mathbb{R}^2 \times [0,1] \to [0,1]$ maps each string homeomorphically onto $[0,1]$ and

$$b \cap (\mathbb{R}^2 \times \{0\}) = \{(1,0,0), (2,0,0), ..., (n,0,0)\}$$
$$b \cap (\mathbb{R}^2 \times \{1\}) = \{(1,0,1), (2,0,1), ..., (n,0,1)\}.$$



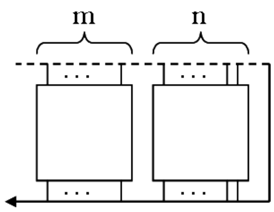
It is natural to identify two geometric braids with the same number of strings if they are isotopic. That means they can continously be deformed into each other via a homotopy which leaves the endpoints of the strings fixed during the process of deformation. The equivalence classes produced via this identification are called *braids*.

Now we turn to the category **Braid** whose objects are by definition all natural numbers $\mathbb{N}$ including 0. The set of morphisms between two objects n and m is the empty set $\emptyset$ unless $n = m$. In the latter case $\mathrm{Hom}_{\mathbf{Braid}}(n,n)$ is defined to be the set of all braids on n strings. If $n = 0$ the set $\mathrm{Hom}_{\mathbf{Braid}}(0,0)$ consists of the *empty braid* $b = \emptyset$ only.

Composition of morphisms is given by concatenation of braids on n strings. Precisely, given two n-string braids we choose two representing geometric braids b_1, b_2 and define their product to be the set of points $(x, y, t) \in \mathbb{R}^2 \times [0, 1]$ such that $(x, y, 2t) \in b_1$ if $0 \leq t \leq \frac{1}{2}$ and $(x, y, 2t-1) \in b_2$ if $\frac{1}{2} \leq t \leq 1$. This yields a new geometric n-string braid and therefore a braid on n-strings. This composition is well-defined and associative. The unit morphism is given by the trivial braid

$$\{1, 2, ..., n\} \times 0 \times [0, 1] \subset \mathbb{R}^2 \times [0, 1].$$

Finally we introduce a monoidal structure in the category **Braid** by defining the tensor product of two objects n, m to be $n \otimes m := n + m$. The unit object is obviously given by 0. The tensor product of two morphisms or more precisely of two braids is realized by placing one braid next to the other.



This concept of paralleling is crucial for this thesis and we will run into it over and over again (cf. Section 2). Since **Braid** is a strict monoidal category we do not have to bother with any of the coherence constraints or diagrams.

Until now we have consistently ignored certain natural symmetries that are present in the categories under consideration. In $\mathbf{Vect}_k$ there is a natural twist isomorphism $\tau_{V,W} : V \otimes W \xrightarrow{\sim} W \otimes V$ given by $x \otimes y \mapsto y \otimes x$ for any particular choice of k-vector spaces V, W. These isomorphisms satisfy certain conditions that are outlined in the following definition.

Definition 9. A *symmetric monoidal category* consists of a monoidal category $(\mathcal{C}, \otimes, a, \mathbf{1}, l, r)$ together with a collection τ of natural isomorphisms

$$\tau_{X,Y} : X \otimes Y \xrightarrow{\sim} Y \otimes X \qquad \forall X, Y \in \mathcal{C}$$

called a *commutativity constraint*. This data is subject to the following axioms:

1. **(Hexagon Axiom)** The diagrams

$$\begin{array}{ccc}
X \otimes (Y \otimes Z) & \xrightarrow{\tau_{X,Y\otimes Z}} & (Y \otimes Z) \otimes X \\
\uparrow{\scriptstyle a_{X,Y,Z}} & & \downarrow{\scriptstyle a_{Y,Z,X}} \\
(X \otimes Y) \otimes Z & & Y \otimes (Z \otimes X) \\
\downarrow{\scriptstyle \tau_{X,Y} \otimes id_Z} & & \uparrow{\scriptstyle id_Y \otimes \tau_{X,Z}} \\
(Y \otimes X) \otimes Z & \xrightarrow{a_{Y,X,Z}} & Y \otimes (X \otimes Z)
\end{array}$$

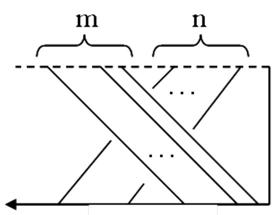
and

$$\begin{array}{ccc} (X \otimes Y) \otimes Z & \xrightarrow{\tau_{X\otimes Y,Z}} & Z \otimes (X \otimes Y) \\ {\scriptstyle a^{-1}_{X,Y,Z}} \uparrow & & \downarrow {\scriptstyle a^{-1}_{Z,X,Y}} \\ X \otimes (Y \otimes Z) & & (Z \otimes X) \otimes Y \\ {\scriptstyle id_X \otimes \tau_{Y,Z}} \downarrow & & \uparrow {\scriptstyle \tau_{X,Z} \otimes id_Y} \\ X \otimes (Z \otimes Y) & \xrightarrow{a^{-1}_{X,Z,Y}} & (X \otimes Z) \otimes Y \end{array}$$

commute $\forall X, Y, Z \in \mathcal{C}$.

2. $\tau_{Y,X} \circ \tau_{X,Y} = id_{X \otimes Y} \quad \forall X, Y \in \mathcal{C}$

If the condition $\tau_{Y,X} \circ \tau_{X,Y} = id_{X \otimes Y}$ in Definition 9 is dropped we obtain what is called a *braided monoidal category*. In the category **Braid** we can define a twist isomorphism $n \otimes m \to m \otimes n$ as illustrated by the following picture



It can be verified that these are natural isomorphisms satisfying the hexagon axioms; see [Kas95, XIII.2]. However, it is geometrically evident that $\tau_{m,n} \circ \tau_{n,m} \neq id_{n \otimes m}$ because applying $\tau_{m,n}$ after $\tau_{n,m}$ twists the braid even further.

After having introduced monoidal categories we pass on to monoidal functors. The following definition categorifies the notion of a monoid homomorphism.

Definition 10. Let $(\mathcal{C}, \otimes, a, \mathbf{1}, l, r)$ and $(\mathcal{C}', \otimes', a', \mathbf{1}', l', r')$ be two monoidal categories. A *monoidal functor* from $\mathcal{C}$ to $\mathcal{C}'$ is given by a triple (F, J, ϕ) where $F : \mathcal{C} \to \mathcal{C}'$ is a functor, J is a natural isomorphism

$$J_{X,Y} : F(X) \otimes F(Y) \xrightarrow{\sim} F(X \otimes Y) \qquad \forall X, Y \in \mathcal{C}$$

and $\phi : \mathbf{1}' \xrightarrow{\sim} F(\mathbf{1})$ is an isomorphism. This data is subject to the following conditions:

1. **(Monoidal Structure Axiom)** The diagram

$$\begin{CD}
(F(X)\otimes' F(Y))\otimes' F(Z) @>{a'_{F(X),F(Y),F(Z)}}>> F(X)\otimes'(F(Y)\otimes' F(Z)) \\
@V{J_{X,Y}\otimes' id_{F(Z)}}VV @VV{id_{F(X)}\otimes' J_{Y,Z}}V \\
F(X\otimes Y)\otimes' F(Z) @. F(X)\otimes' F(Y\otimes Z) \\
@V{J_{X\otimes Y,Z}}VV @VV{J_{X,Y\otimes Z}}V \\
F((X\otimes Y)\otimes Z) @>{F(a_{X,Y,Z})}>> F(X\otimes(Y\otimes Z))
\end{CD}$$

is commutative $\forall X,Y,Z\in\mathcal{C}$.

2. The diagrams

$$\begin{CD}
\mathbf{1}'\otimes' F(X) @>{l'_{F(X)}}>> F(X) \\
@V{\phi\otimes' id_{F(X)}}VV @AA{F(l_X)}A \\
F(\mathbf{1})\otimes' F(X) @>{J_{\mathbf{1},X}}>> F(\mathbf{1}\otimes X)
\end{CD}
\qquad
\begin{CD}
F(X)\otimes'\mathbf{1}' @>{r'_{F(X)}}>> F(X) \\
@V{id_{F(X)}\otimes'\phi}VV @AA{F(r_X)}A \\
F(X)\otimes' F(\mathbf{1}) @>{J_{X,\mathbf{1}}}>> F(X\otimes\mathbf{1})
\end{CD}$$

are commutative $\forall X\in\mathcal{C}$.

A monoidal functor is called *strict* if the isomorphisms $J_{X,Y}$ and ϕ are identities.

Definition 11. Let $(\mathcal{C},\otimes,a,\mathbf{1},l,r)$ and $(\mathcal{C}',\otimes',a',\mathbf{1}',l',r')$ be monoidal categories and (F,J,ϕ), $(\tilde{F},\tilde{J},\tilde{\phi})$ two monoidal functors from $\mathcal{C}$ to $\mathcal{C}'$. A *natural monoidal transformation* $\eta:(F,J,\phi)\to(\tilde{F},\tilde{J},\tilde{\phi})$ is a natural transformation $\eta:F\to\tilde{F}$ such that the following diagrams commute for all $X,Y\in\mathcal{C}$

$$\begin{CD}
F(X)\otimes' F(Y) @>{J_{X,Y}}>> F(X\otimes Y) \\
@V{\eta_X\otimes'\eta_Y}VV @VV{\eta_{X\otimes Y}}V \\
\tilde{F}(X)\otimes'\tilde{F}(Y) @>{\tilde{J}_{X,Y}}>> \tilde{F}(X\otimes Y)
\end{CD}
\qquad
\begin{array}{ccc}
\mathbf{1}' & \xrightarrow{\ \phi\ } & F(\mathbf{1}) \\
 & \underset{\tilde{\phi}}{\searrow} & \downarrow{\scriptstyle \eta_{\mathbf{1}}} \\
 & & \tilde{F}(\mathbf{1})
\end{array}$$

A *natural monoidal isomorphism* is a natural monoidal transformation which is also a natural isomorphism.

Now that we have introduced all notions necessary to understand the definition of a TQFT (cf. Definition 1) we close this introductory section by mentioning MacLane's strictness theorem.

Theorem 12. *Every monoidal category is monoidally equivalent to a strict monoidal category. More precisely, given a monoidal category $\mathcal{C}$ there exists a strict monoidal category $\mathcal{C}_{str}$ together with monoidal functors $F : \mathcal{C} \to \mathcal{C}_{str}$ and $F' : \mathcal{C}_{str} \to \mathcal{C}$ such that we have natural monoidal isomorphisms $FF' \cong id_{\mathcal{C}_{str}}$ and $F'F \cong id_{\mathcal{C}}$.*

Proof. For more information and a proof of this important result consult [Kas95, XI.5] or [ML98, XI.3]. □

Viewing equivalent categories as essentially the same we will use this theorem to work with strict categories whenever we want to.

2 Cobordisms

Now that TQFTs and monoidal categories have been introduced in full generality this section devotes itself to one single but significant example of a symmetric monoidal category, namely the category of n-dimensional cobordisms **nCob**. As mentioned in the introduction this category is the classical domain category of a TQFT.

The objective of this section is as follows. After introducing cobordisms and explaining the category **nCob** in the first two parts we will then limit ourselves to **2Cob**. It turns out that the 2-dimensional case can completely be understood by providing a presentation of the category **2Cob**. This will be the key to understand the connection between 2d-TQFTs and Frobenius algebras.

Since this thesis highlights algebraic aspects of TQFTs rather than differential topology we will not prove everything in detail. The given sources contain much further information. Our presentation of the material is basically a condensed version of [Koc04, pp.9-77]. Basic knowledge about notions related to smooth manifolds will be assumed, see e.g. [Lee02].

2.1 Oriented cobordisms

Let M be a compact oriented n-manifold[4] with boundary. In particular this means that every point $x \in M$ has an open neighborhood $U \subset M$ which is homeomorphic to an open subset of the half-space $\mathbb{H} := \{(x_1, ..., x_n) \in \mathbb{R}^n \,|\, x_n \geq 0\}$. The boundary ∂M is again an orientable manifold of dimension $n-1$ where the orientation of ∂M is normally chosen to be induced by the orientation of M.

Now focus on a connected component Σ of ∂M. We could also define an orientation of the manifold Σ independently of the orientation of M. The reason to consider this is that the boundary components can be characterized as in- or out-boundary components by specifying another orientation at will.

[4]In this thesis all manifolds are supposed to be equipped with a smooth structure.

Let $x \in \Sigma$ be a point and v_1,...,v_{n-1} be a positively oriented basis of $T_x\Sigma$. Since the tangent bundle $T\Sigma$ can be thought of as a subbundle of $TM \mid_\Sigma$ we can take a vector $w \in T_xM$ and ask whether the set v_1,...,v_{n-1},w defines a positively oriented basis of T_xM. If this is the case we will refer to w as a *positive normal*.

Recall from the definition of the tangent space that w is just an equivalence class of curves passing through $x \in M$. In particular we could take a chart ϕ around x and locally view a representing curve as a curve in $\mathbb{R}^n$. Now it is sensible to ask wether the tangent vector of this curve at $\phi(x)$ points into the half-space. If this is the case we simply say that w is *inward-pointing*.

Definition 13. A connected component $\Sigma \subset \partial M$ is called an *in-boundary* component if for some $x \in \Sigma$ a positive normal is inward-pointing. Otherwise it is called an *out-boundary* component.

It can be checked that this is well-defined in the sense that the definition is neither dependent on any particular choice of $x \in \Sigma$ nor on the choice of a positive normal. Therefore the boundary of a manifold M consists of certain in- and out-boundary components.

Definition 14. Let Σ_0 and Σ_1 be closed oriented $(n-1)$-manifolds. An *oriented cobordism* $M : \Sigma_0 \Rightarrow \Sigma_1$ from Σ_0 to Σ_1 is a compact oriented n-manifold M together with smooth maps $\Sigma_0 \to M$ and $\Sigma_1 \to M$ such that Σ_0 maps diffeomorphically, preserving orientation, onto the in-boundary of M, and Σ_1 maps diffeomorphically, preserving orientation, onto the out-boundary of M.

Remark 15 (Cylinder construction)**.** To illustrate this concept we discuss a crucial construction which produces important examples of oriented cobordisms. Let Σ_0 and Σ_1 be two closed oriented $(n-1)$-manifolds which are diffeomorphic via an orientation-preserving diffeomorphism. We define an oriented n-manifold by $M := \Sigma_1 \times [0,1]$ where $[0,1]$ is equipped with its canonical orientation and so is M. Then M together with the smooth maps

$$\Sigma_0 \xrightarrow{\sim} \Sigma_1 \xrightarrow{\sim} \Sigma_1 \times \{0\} \hookrightarrow \Sigma_1 \times [0,1]$$

and

$$\Sigma_1 \xrightarrow{\sim} \Sigma_1 \times \{1\} \hookrightarrow \Sigma_1 \times [0,1]$$

constitutes a cobordism from Σ_0 to Σ_1 because $\Sigma_1 \times \{0\}$ is the inboundary of M and $\Sigma_1 \times \{1\}$ is the outboundary. Thus we have found a way to assign a cobordism to a pair of manifolds that come together with an orientation preserving diffeomorphism. This is called the *cylinder construction*.

2.2 The category nCob

The next step would be to define a category whose objects are closed oriented $(n-1)$-manifolds and whose morphisms are oriented cobordisms. Fortunately it turns out that the natural way of doing this is impossible[5]. We quickly demonstrate where the construction fails and see how the problem is solved. Afterwards we are ready to define the right version of **nCob**.

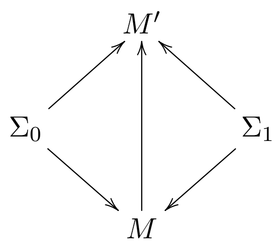
In a category where oriented cobordisms are morphisms we might naively define a composition as follows. Let Σ_0, Σ_1 and Σ_2 be closed oriented $(n-1)$-manifolds and let $M_0 : \Sigma_0 \Rightarrow \Sigma_1$ as well as $M_1 : \Sigma_1 \Rightarrow \Sigma_2$ be two oriented cobordisms. We can then glue these manifolds along their common boundary Σ_1 as topological manifolds. The following theorem asserts that the glued topological manifold $M_0M_1 := M_0 \amalg_{\Sigma_1} M_1$ can be equipped with a smooth structure again.

Theorem 16. *Given two cobordisms $M_0 : \Sigma_0 \Rightarrow \Sigma_1$ and $M_1 : \Sigma_1 \Rightarrow \Sigma_2$ there exists a smooth structure on the topological manifold $M_0M_1 = M_0 \amalg_\Sigma M_1$ such that each inclusion map $M_0 \hookrightarrow M_0M_1$, $M_1 \hookrightarrow M_0M_1$ is a diffeomorphism onto its image. This smooth structure is unique up to a diffeomorphism leaving Σ_0, Σ_1 and Σ_2 fixed.*

Proof. The proof requires Morse theory. For details see [Mil65, cf. Thm.1.4]. □

The theorem suggests that we have to be careful. The problem is that in general there is no canonical choice of a smooth structure on the glued manifold. In other words the composition of oriented cobordisms described above is not well-defined. Luckily this problem of uniqueness can be solved by introducing an equivalence relation between oriented cobordisms.

Definition 17. Two cobordisms from Σ_0 to Σ_1 are *equivalent* if there exists an orientation-preserving diffeomorphism $\psi : M \to M'$ such that the following diagram commutes



[5]This is not a typo. The fortunate part about this problem is that it quickly initiated the search for an alternative way of defining a category of oriented cobordisms which at first sight seems less intuitive (since it involves the introduction of an equivalence relation) but on the other hand made the construction of manifold invariants possible in the first place.

Now the idea is to compose cobordism classes rather than cobordisms themselves. Again let Σ_0, Σ_1 and Σ_2 be closed oriented $n-1$-manifolds and let $M_0 : \Sigma_0 \Rightarrow \Sigma_1$ and $M_1 : \Sigma_1 \Rightarrow \Sigma_2$ be representatives of the cobordism classes $[M_0]$ and $[M_1]$ respectively. By Theorem 16 the manifold $M_0 M_1$ represents a well-defined cobordism class. It can be checked that the class $[M_0 M_1]$ only depends on the classes $[M_0]$ and $[M_1]$ and not on the choice of representatives. Thus we have a well-defined notion of composing cobordism classes which turns out to be associative because gluing of manifolds is associative.

In order to define an identity cobordism class for a given Σ we come back to the construction in Remark 15. There we saw that an orientation-preserving diffeomorphism of $(n-1)$-manifolds induces an oriented cobordism between these manifolds. Choosing both Σ_0 and Σ_1 to be Σ and the diffeomorphism to be the identity map we obtain an oriented cobordism $\Sigma \Rightarrow \Sigma$ whose class serves as the identity cobordism. For a Morse-theoretical proof see [Koc04, 1.3.16]. Thus we have a category.

Definition 18. By **nCob** we denote the category whose objects are closed oriented $(n-1)$-manifolds and whose morphisms are classes of oriented cobordisms as described above.

The monoidal structure in this category is very similar to the one in **Braid** (cf. Example 8). Given two closed oriented $(n-1)$-manifolds their tensor product is defined to be their disjoint union which is again a closed oriented $(n-1)$-manifold. Analogously, the tensor product of two cobordism classes is given by the class of the oriented cobordism obtained from taking the disjoint union of a representing manifold from each class. The unit object is given by the empty manifold.

To define a symmetric structure in this category we use the cylinder construction again. For given $(n-1)$-manifolds Σ_0 and Σ_1 with the usual properties the twist isomorphism $\tau_{\Sigma_0,\Sigma_1} : \Sigma_0 \amalg \Sigma_1 \Rightarrow \Sigma_1 \amalg \Sigma_0$ is defined to be the class of the cobordism induced by the canonical twist diffeomorphism $\Sigma_0 \amalg \Sigma_1 \to \Sigma_1 \amalg \Sigma_0$.

At this point it is a good place to quickly convince oneself that the cylinder construction is not only a simple assignment but even a functor from the category of closed oriented $(n-1)$-manifolds with orientation-preserving diffeomorphisms to the category **nCob**, see [Koc04, 1.3.22]. This insight immediately implies that τ_{Σ_0,Σ_1} is truly an isomorphism in **nCob** and moreover that $\tau_{\Sigma_1,\Sigma_0} \circ \tau_{\Sigma_0,\Sigma_1} = id_{\Sigma_0,\Sigma_1}$. The rest of the defining properties of a symmetric structure (in particular the naturality of the isomorphisms) can be deduced exploiting the fact that the twist diffeomorphism $\Sigma_0 \amalg \Sigma_1 \to \Sigma_1 \amalg \Sigma_0$ turns the category of smooth manifolds into a symmetric monoidal category. For now we will not bother with that in any detail because it turns out that in the case of our main interest $(n = 2)$ all these things will be rather obvious.

2.3 A presentation of 2Cob

Finally we want to focus on the category **2Cob** since we get an explicit description of this category in terms of generators and relations. As it turns out this will be the key to translating all the topological data into algebra.

We begin by replacing the category **2Cob** with a category which is equivalent but somewhat simpler. This category will be a skeleton of **2Cob**.

Lemma 19. *For $n \geq 0$ let $\mathbf{n}$ denote the disjoint union of n copies of the circle S^1 with some fixed orientation. By $\mathbf{0}$ we denote the empty 1-manifold $\emptyset$. Then the full subcategory consisting of objects $\{\mathbf{0}, \mathbf{1}, \mathbf{2}, ...\}$ is a skeleton of* **2Cob**.

Proof. It is a well-known result that any closed 1-manifold is diffeomorphic to a finite union of copies of S^1, cf. [Mil97, Appendix]. In fact any closed oriented 1-manifold is diffeomorphic via an orientation preserving diffeomorphism to a finite union of copies of S^1 with fixed orientation. This follows from the existence of an orientation preserving diffeomorphism between two copies of S^1 with reverse orientation.[6] Thus it is enough to show that two objects are isomorphic in **2Cob** if and only if they are diffeomorphic as manifolds via an orientation-preserving diffeomorphism. Such a diffeomorphism induces an isomorphism in **2Cob** because we have already noted above that the cylinder construction is functorial. On the other hand the existence of an isomorphism between two closed oriented 1-manifolds in **2Cob** implies that both manifolds share the same number of connected components, see [Koc04, 1.3.30]. Then, by the argumentation above, there exists an orientation-preserving diffeomorphism between these manifolds. □

Notice that this skeleton is obviously closed under the operation of disjoint union and thus the monoidal structure carries over to this category. From now on and throughout this thesis we will write **2Cob** for the skeleton of the original **2Cob**.

The following result lays the foundation for our further discussion.

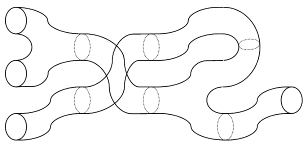
Theorem 20. *Two connected, compact, oriented surfaces are diffeomorphic if and only if they have the same genus and the same number of boundary components.*

Proof. [Hir76, Thm.3.11] □

This theorem allows us to classify two-dimensional oriented cobordism classes via the genus of a representing surface as long as we additionally keep

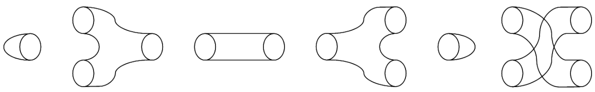
[6]This diffeomorphism can be constructed by placing the two circles side by side in a plane, separated by a vertial line of equal distance from both circles. Then the two circles can be thought of as mirror images of each other by reflection in the line. Mapping points to their mirror images yields the desired diffeomorphism.

track of which boundary components are in and which are out. In particular a picture of a cobordism like this

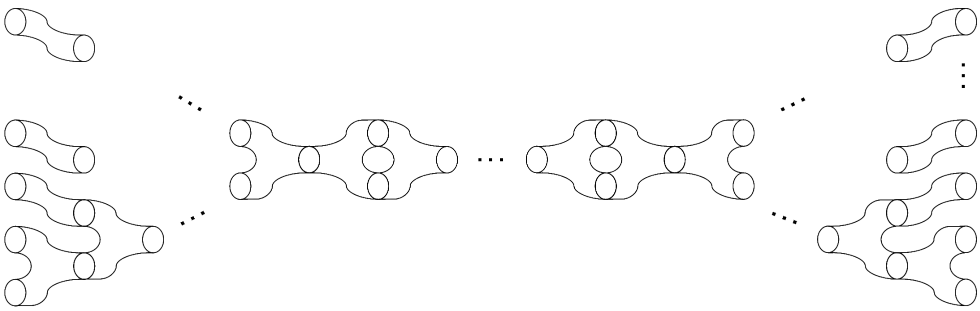


uniquely describes a certain cobordism class because it contains all necessary information about the genus as well as the in- and out-boundary components (we follow the convention that these pictures are read from bottom to top and all in-boundary components are drawn on the left). In this case the picture describes the class of a cobordism $\mathbf{3} \Rightarrow \mathbf{1}$ of genus 0. Even though one might guess from the picture that the surface penetrates itself in the middle this is not what is meant by this drawing. It rather symbolizes the fact that our manifolds are not embedded in an ambient space and thus we simply do not know which component lies above which. In fact the notion of "over" and "under" does not even exist.

Theorem 21. *Any cobordism class in the monoidal category* **2Cob** *can be obtained by either composing or paralleling (disjoint union) the classes of the following six elementary cobordisms*



Proof. We begin by looking at a cobordism class represented by a connected surface. By the classification theorem of surfaces one can immediately write down a canonical representative made up of the given basic cobordisms via the following normal form



This normal form is made up of three parts. The first one being a cobordism $\mathbf{n} \Rightarrow \mathbf{1}$ encoding the data regarding the in-boundary. The second one contains the topological data in form of the genus. The last part is a cobordism $\mathbf{1} \Rightarrow \mathbf{m}$ describing the out-boundary part. If the in-part is a cobordism $\mathbf{0} \Rightarrow \mathbf{1}$ the first elementary cobordism listed above is used to construct the normal form. Similarly the fifth elementary cobordism is used if the out-part is a cobordism $\mathbf{1} \Rightarrow \mathbf{0}$.

If the representing surface is not connected we have to be a bit careful since disjoint union in the category of manifolds is not the same as disjoint union in **2Cob** simply because equivalent cobordisms have to respect a specific ordering of the in- and out-boundary components. So for disconnected cobordisms we could use the normal form described above for each connected component and afterwards permute the in- and out-boundary components until they fit the cobordism class which we like to represent. Since any permutation can be decomposed as a product of neighboring transpositions the elementary twist suffices to do that. □

Having found a set of generators for **2Cob** we ask about relations. For the sake of readability these relations are listed in Appendix B. The proof of these relations is trivial having the classification result in mind. For each relation simply notice that each of the surfaces involved has genus zero and the same number of in- and out-boundary components which respect the ordering. In particular the twist relations listed there show that **2Cob** is a symmetric monoidal category.

Finally we state that the relations listed in Appendix B are in fact sufficient in the sense that every other relation that someone might write down can be obtained by building it from the relations that are already listed. In other words the relations are sufficient to transform any given decomposition of a cobordism to normal form. We will not address this issue any further. For details consult [Koc04, p.73-77].

3 Frobenius algebras

The following constitutes a core section of this thesis. The reason for this establishes itself in the fact that 2d-TQFTs can be characterized by Frobenius algebras (cf. Theorem 37). More specific a 2d-TQFT corresponds to a commutative Frobenius algebra and vice versa.

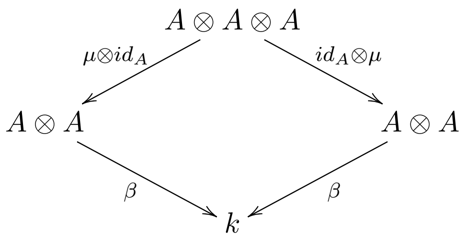
In the first part we will introduce the notion of a Frobenius algebra and give some important examples. Afterwards we want to show that under certain assumptions a Frobenius algebra admits a unique coalgebra structure whose counit is the Frobenius form. Theorem 36 will make this useful statement precise.

3.1 Definition of a Frobenius algebra

All of the basics on Frobenius algebras presented in the following are standard, see e.g. [Abr97] or [Koc04]. However, the formulations and proofs given here may differ slightly since they have been adapted to the needs of this thesis. For simplicity we will work with a strictified version of $\mathbf{Vect}_k$ and identify $A\otimes k = A = k\otimes A$ as well as $A\otimes(A\otimes A) = (A\otimes A)\otimes A = A\otimes A\otimes A$ (cf. Section 1).

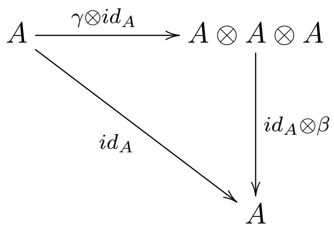
Definition 22. Let A be a k-algebra[7] with multiplication map μ and unit η. An *associative, non-degenerate pairing* is a k-linear map $\beta : A \otimes A \to k$ such that

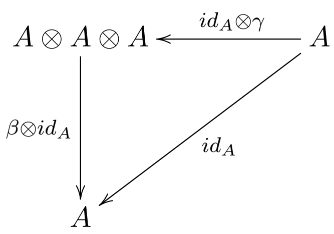
1. The diagram



   is commutative.

2. There exists a k-linear map $\gamma : k \to A \otimes A$ such that





   are commutative. The map γ is called a *copairing.*

Lemma 23. *Let A be a finite-dimensional k-algebra. There is a bijection between*

1. *k-linear maps $\varepsilon : A \to k$ such that $\varepsilon(\mu(a \otimes b)) = 0$ for all $a \in A$ implies $b = 0$*

2. *associative, non-degenerate pairings $\beta : A \otimes A \to k$.*

A k-linear map $\varepsilon : A \to k$ with the properties described above is called a Frobenius form.

[7]In this text we use a definition of a k-algebra which is common in the theory of quantum groups. Please consult Appendix A for further information.

Proof. Given a linear map $\varepsilon : A \to k$ with the property of the lemma we define a pairing β as follows

$$\beta : A \otimes A \longrightarrow k\,, \quad a \otimes b \longmapsto \varepsilon(\mu(a \otimes b)).$$

Since μ is associative we have

$$\mu \circ \mu \otimes id_A = \mu \circ id_A \otimes \mu.$$

Precomposing with ε gives the associativity of $\beta = \varepsilon \circ \mu$.
In order to see the non-degeneracy we will explicitly construct a copairing γ. To do that consider the k-linear map

$$f : A \to A^*\,, \quad b \mapsto \varepsilon(\mu(\_ \otimes b))$$

from A to its vector space dual. Notice that f is injective: Let $f(b) = 0$ for some $b \in A$. This means $\varepsilon(\mu(\_ \otimes b))$ is the zero map. Explicitly we have $\varepsilon(\mu(a \otimes b)) = 0$ for all $a \in A$. Hence by the properties of ε we get $b = 0$, which shows injectivity.
In particular, if we choose a basis b_1,...,b_n of A, injectivity of f implies that the linear forms $\varepsilon(\mu(\_ \otimes b_i))$ constitute a basis of A^*. Now it is easy to verify that the matrix $(b_{ij})_{ij}$ with $b_{ij} := \beta(b_i \otimes b_j)$ is invertible.
Let $(\gamma_{ij})_{ij}$ be its inverse. Now define

$$\gamma : k \to A \otimes A\,, \quad 1 \mapsto \sum_{i,j=1}^{n} \gamma_{ij}.b_i \otimes b_j.$$

This satisfies the commutative diagrams expressing non-degeneracy. By linearity it suffices to show this on a basis vector b_k.

$$\begin{aligned} id \otimes \beta \circ \gamma \otimes id(b_k) &= id \otimes \beta(\sum_{i,j=1}^{n} \gamma_{ij}.b_i \otimes b_j \otimes b_k) \\ &= \sum_{i,j=1}^{n} \gamma_{ij}\beta_{jk}.b_i \\ &= \sum_{i=1}^{n}(\sum_{j=1}^{n} \gamma_{ij}\beta_{jk}).b_i \\ &= b_k \end{aligned}$$

The last equation used the fact that $\sum_{j=1}^{n} \gamma_{ij}\beta_{jk} = \delta_{ik}$ where δ_{ik} denotes the Kronecker delta. Analogously we compute

$$\beta \otimes id \circ id \otimes \gamma(b_k) = b_k.$$

Hence β is an associative, non-degenerate pairing.

Vice versa suppose we are given an associative, non-degenerate pairing $\beta : A \otimes A \to k$. This defines a linear map by setting

$$\varepsilon : A \to k\,, \quad a \mapsto \beta(a \otimes \eta(1_k)).$$

Fix $b \in A$ and let $\varepsilon(\mu(a \otimes b)) = 0$ for all $a \in A$. By the associativity of β and the properties of the unit we obtain

$$0 = \beta(\mu(a \otimes b) \otimes \eta(1_k)) = \beta(a \otimes \mu(b \otimes \eta(1_k))) = \beta(a \otimes b)$$

for all $a \in A$ and thus

$$\begin{aligned} b &= (id \otimes \beta \circ \gamma \otimes id)(b) \\ &= (id \otimes \beta)(\gamma(1_k) \otimes b) \\ &= (id \otimes \beta)(\sum_{i,j=1}^{n} \gamma_{ij}.b_i \otimes b_j \otimes b) \\ &= \sum_{i,j=1}^{n} \gamma_{ij} \underbrace{\beta(b_j \otimes b)}_{=0} .b_i \\ &= 0. \end{aligned}$$

Finally, we convince ourselves that the assignments defined above are in fact inverse to each other. If we start with a pairing β, go over to the associated linear form ε given by $a \mapsto \beta(a \otimes \eta(1_k))$ and then go back, we find that the pairing obtained from this is given by $a \otimes b \mapsto \beta(\mu(a \otimes b) \otimes \eta(1_k))$. As calculated above we have

$$\beta(\mu(a \otimes b) \otimes \eta(1_k)) = \beta(a \otimes \mu(b \otimes \eta(1_k)) = \beta(a \otimes b).$$

Hence we see that our original β is recovered. The other direction simply follows from

$$\varepsilon(\mu(a \otimes \eta(1_k))) = \varepsilon(a).$$

□

As a consequence of this proof we note the following result which turns out to be needful later.

Corollary 24. *Given an associative, non-degenerate pairing $\beta : A \otimes A \to k$, then the corresponding copairing $\gamma : k \to A \otimes A$ is unique.*

Proof. Let γ and $\tilde{\gamma}$ be two copairings defined by

$$\gamma(1_k) = \sum_{i,j=1}^{n} \gamma_{ij}.b_i \otimes b_j$$

and similarly

$$\tilde{\gamma}(1_k) = \sum_{i,j=1}^{n} \tilde{\gamma}_{ij}.b_i \otimes b_j$$

where b_1,...,b_n is a basis of A. By the commutativity of the diagram expressing non-degeneracy we get

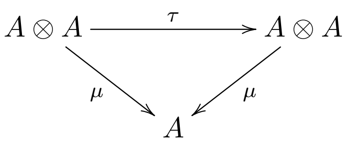
$$id \otimes \beta \circ \gamma \otimes id(b_k) = \sum_{i=1}^{n}(\sum_{j=1}^{n} \gamma_{ij}\beta_{jk}).b_i = b_k$$

and

$$id \otimes \beta \circ \tilde{\gamma} \otimes id(b_k) = \sum_{i=1}^{n}(\sum_{j=1}^{n} \tilde{\gamma}_{ij}\beta_{jk}).b_i = b_k$$

as calculated in the proof of Lemma 23 above. These equations show that both the matrix (γ_{ij}) and $(\tilde{\gamma}_{ij})$ are inverses of the matrix (β_{ij}) and thus they are the same. Hence the two copairings agree. □

Definition 25. A finite-dimensional k-algebra A together with a Frobenius form is called *Frobenius algebra*. A is called *commutative Frobenius algebra* if additionally the following diagram commutes



where $\tau : A \otimes A \to A \otimes A$ is given by the flip $x \otimes y \mapsto y \otimes x$.

Remark 26. Due to Lemma 23 we could equivalently define a Frobenius algebra to be a finite-dimensional k-algebra equipped with an associative, non-degenerate pairing because the bijection between such pairings and Frobenius forms allows us to switch from one to the other whenever it seems convenient. If a Frobenius algebra is specified by an associative, non-degenerate pairing we will refer to it as a *Frobenius pairing.*

In the following we want to study some examples of Frobenius algebras. A long list of examples is provided by [Koc04, p.99 ff.]. The first two discussed below can be found there.

Example 27. Let k be a field. Then we can view k as an algebra over itself. As a Frobenius form we simply take the identity. Since k has no zero divisors this is a well-defined Frobenius form which turns k into a Frobenius algebra.

Example 28. Consider the group algebra $\mathbb{C}[G] = \{\sum_{i=0}^{n} \lambda_i x_i \mid \lambda_i \in \mathbb{C}\}$ over $\mathbb{C}$ of a finite group $G = \{x_0, x_1, ..., x_n\}$ of order $n+1$. Let $x_0 = 1_G$ denote the neutral element in G. Then the group algebra $\mathbb{C}[G]$ becomes a Frobenius algebra by defining the Frobenius form ε to be the linear extension of the map $G \to \mathbb{C}$ given by

$$x_i \mapsto \begin{cases} 1, & \text{for } i = 0 \\ 0, & \text{otherwise.} \end{cases}$$

Now fix some $h = \sum_{i=0}^{n} \lambda_i x_i \in \mathbb{C}[G]$. It remains to check that $\varepsilon(g \cdot h) = 0$ for all $g \in \mathbb{C}[G]$ implies $h = 0$ (cf. Lemma 23). Under the assumption that $\varepsilon(g \cdot (\sum_{i=0}^{n} \lambda_i x_i)) = 0$ for all $g \in \mathbb{C}[G]$, we can choose $g = x_j$ for some $0 \le j \le n$. Let x_k denote the inverse of x_j in G. Then we obtain

$$0 = \varepsilon(x_j \cdot (\sum_{i=0}^{n} \lambda_i x_i)) = \varepsilon(\sum_{i=0}^{n} \lambda_i (x_j x_i)) = \lambda_k.$$

Repeating this for all j yields $\sum_{i=0}^{n} \lambda_i x_i = 0$.

Example 29. The following example requires some fundamental theorems from algebraic topology. All definitions and theorems used here can be found in [Hat03, chapter 3]. The idea of the following example is sketched by Abrams [Abr97, p.58-59]. However, he remains silent about many aspects, e.g. the problem of finite dimensionality, which is one of the defining properties of a Frobenius algebra. Here we present a more detailed and worked out version.

Let M be a smooth, compact, k-oriented n-manifold and k is some field of characteristic zero. We would like to equip the (singular) cohomology ring

$$H^*(M) := \bigoplus_{i \ge 0} H^i(M; k)$$

with coefficients in k with the structure of a Frobenius algebra.

The first thing to notice here is that $H^*(M)$ is not only a ring but even a k-algebra since all the the cohomology groups $H^i(M,k)$ are in fact vector spaces. Furthermore $H^*(M)$ is finite-dimensional. The crucial result to see this is that a n-dimensional manifold has the homotopy type of a CW-complex of dimension less than or equal to n [Hir76, Thm.4.3]. Cellular cohomology shows that $H^i(M) = 0$ for $i > n$. Moreover, the compactness of M implies that M is indeed a finite CW-complex, that is M is made up of only finitely many cells. Thus we conclude the finite-dimensionality of $H^*(M) = \bigoplus_{i=0}^{n} H^i(M)$.

It remains to construct a Frobenius form. To do that consider the map

$$\langle -, - \rangle : H^i(M) \otimes H_i(M) \to k, \qquad [f] \otimes [c] \mapsto \langle [f], [c] \rangle := f(c).$$

It is easy to check that for each i this is a well-defined evaluation map and does not depend on the choice of any representative. Using this we define a

k-linear map

$$\varepsilon_n : H^n(M) \to k, \qquad [f] \mapsto \langle [f], [M] \rangle$$

where $[M]$ denotes the fundamental orientation class of M. For $0 \leq i < n$ we define

$$\varepsilon_i : H^i(M) \to k$$

to be the zero map. By the universal property of the direct sum we obtain a k-linear map

$$\varepsilon : \bigoplus_{i=0}^{n} H^i(M; k) \to k$$

which is given by $\langle [f], [M] \rangle$ for $[f] \in H^n(M)$ and zero otherwise. The claim is that ε is a Frobenius form. Since the multiplication map $\mu : H^*(M) \otimes H^*(M) \to H^*(M)$ is given by the cup-product

$$[f] \otimes [g] \mapsto [f] \cup [g]$$

we have to show that $\varepsilon([f] \cup [g]) = 0$ for all $[f] \in H^*(M)$ implies that $[g]$ must be zero.

To prove this we first introduce some useful isomorphisms.[8] Since our manifold is compact and k-orientable we have the Poincare duality isomorphism

$$H^i(M) \xrightarrow{\sim} H_{n-i}(M), \qquad [f] \mapsto [f] \cap [M]$$

where $\cap$ denotes the cap-product. Moreover, there is the canonical isomorphism from $H_{n-i}(M)$ to its double dual $\mathrm{Hom}(\mathrm{Hom}(H_{n-i}(M), k), k)$ which is explicitly given by

$$[c] \mapsto ((\varphi : H_{n-i}(M) \to k) \mapsto \varphi([c])).$$

By the universal coefficient theorem we have an isomorphism

$$H^{n-i}(M) \xrightarrow{\sim} \mathrm{Hom}(H_{n-i}(M), k), \qquad [f] \mapsto \langle [f], - \rangle$$

since $\mathrm{Ext}_k(H_{n-i-1}(M), k) = 0$, because k is a field. Applying the Hom-functor yields an isomorphism

$$\mathrm{Hom}(\mathrm{Hom}(H_{n-i}(M), k), k) \xrightarrow{\sim} \mathrm{Hom}(H^{n-i}(M), k)$$

which is simply precomposing with the universal coefficient theorem isomorphism.

To sum up, we have an isomorphism $H^i(M) \xrightarrow{\sim} \mathrm{Hom}(H^{n-i}(M), k)$ by the composition

$$H^i(M) \xrightarrow{\sim} H_{n-i}(M) \xrightarrow{\sim} \mathrm{Hom}(\mathrm{Hom}(H_{n-i}(M), k), k) \xrightarrow{\sim} \mathrm{Hom}(H^{n-i}(M), k)$$

[8]In this example we will use the shorthand notation $\mathrm{Hom}(V, W)$ whenever we actually mean $\mathrm{Hom}_k(V, W)$, the space of all k-linear maps between some k-vector spaces V, W.

which is explicitly given by

$$[f] \mapsto [f] \cap [M] \mapsto ((\varphi : H_{n-i}(M) \to k) \mapsto \varphi([f] \cap [M])) \mapsto \langle -, [f] \cap [M] \rangle.$$

After these preliminaries we can finally prove that the linear form ε is indeed a Frobenius form. So let $\varepsilon([f] \cup [g]) = 0$ for all $[f] \in H^*(M)$ and $[g] \in H^i(M)$ fixed. Obviously the interesting case is $[f] \in H^{n-i}(M)$ such that $[f] \cup [g] \in H^n(M)$. Then we have

$$\begin{aligned} 0 = \varepsilon([f] \cup [g]) &= \langle [f] \cup [g], [M] \rangle \\ &= (-1)^{(n-i) \cdot i} \langle [f], [g] \cap [M] \rangle \end{aligned}$$

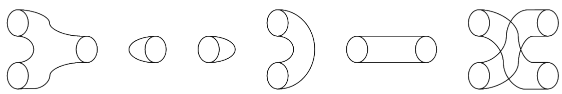
for all $[f] \in H^{n-i}(M)$. Thus $\langle -, [g] \cap [M] \rangle$ is the zero map. The isomorphism $H^i(M) \xrightarrow{\sim} \mathrm{Hom}(H^{n-i}(M), k)$ shows that $[g] = 0$ which is exactly what we wanted.

3.2 Frobenius algebras and coalgebras

The aim of this section is to see that a Frobenius algebra carries a coalgebra structure whose counit is the Frobenius form. This coalgebra structure turns out to be unique if one requires the Frobenius relation to hold (cf. Theorem 36). To construct the comultiplication map δ a graphical calculus is provided which replaces the work with commutative diagrams. This will already anticipate our main classification theorem since our pictures will resemble two-dimensional cobordisms. The idea of using this graphical calculus was inspired by Kock's book [Koc04, 2.3]. All of the results can be found in his text. However, we managed to shorten his exposition. For example Kock introduces a three-point function in order to define the comultiplication. Despite the importance that this might have for field theory we decided to take a shortcut since this function is not needed anywhere.

3.2.1 Graphical calculus

If we start with a Frobenius algebra A we are given maps μ, η, ε, β and the flip τ. We will now represent each of these maps by a symbol as shown below. Since the identity map id_A occurs in the diagrams expressing the properties of these maps as well, we will adopt a symbol for it, too.



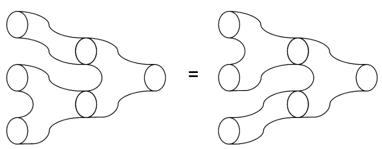
μ η ε β id_A τ

Remark 30. The idea behind these symbols is the following: We count the tensor powers occuring in the source of each map and draw a circle for each power on the left side of our picture. Similarly we draw a circle for each tensor power in the target on the right side of the picture and join these circles by something which resembles a surface. Note that whenever the ground field k appears in our maps we interpret this as the zeroth tensor power of A. Hence, according to our principles, we do not draw a circle for it. Moreover, these pictures are supposed to be read from bottom to top. In other words the first algebra occuring in a tensor power is represented by the lowest circle.

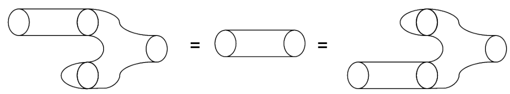
Since each of the symbols described above actually stands for a map, it is natural that we want to have something which respresents composition and tensoring of maps graphically. Thus we introduce the following set of rules for this graphical calculus

1. Taking a tensor product of two maps is graphically represented by simply putting the symbol representing the second map in the tensor product on top of the first one.
2. Composition of maps is symbolized by joining the circles in the picture of the first map on the right side with the circles in the picture on the left side of the second map.

In order to familiarize ourselves with this graphical calculus we will express the commutative diagrams in the definition of a k-algebra (cf. Appendix A) in terms of the pictures. We will need this later anyway.

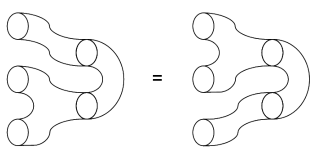


Associativity (of μ)

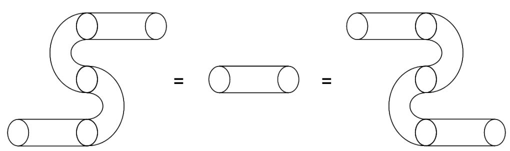


Unit Axioms

In order to get a full graphical description of a Frobenius algebra we also have to express the conditions imposed on the Frobenius form in terms of our pictures. Obviously this is hardly possible because here we resort to dealing with elements explicitly which our calculus is not capable of. So we would rather like to work with a Frobenius pairing. Then Definition 22 gives the following pictures



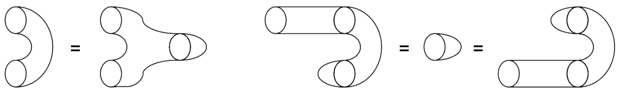
Associativity (of β)



Non-degeneracy / Snake relation

where the turned pairing obviously stands for the copairing.

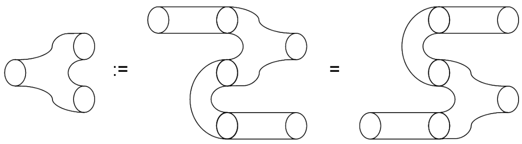
Recall from the proof of Lemma 23 that there is a direct connection between ε and β. In particular we got equalities $\varepsilon \circ \mu = \beta$, $\mu \circ \eta \otimes id = \varepsilon$ and $\mu \circ id \otimes \eta = \varepsilon$. Since this will be important later we write these relations graphically, too.



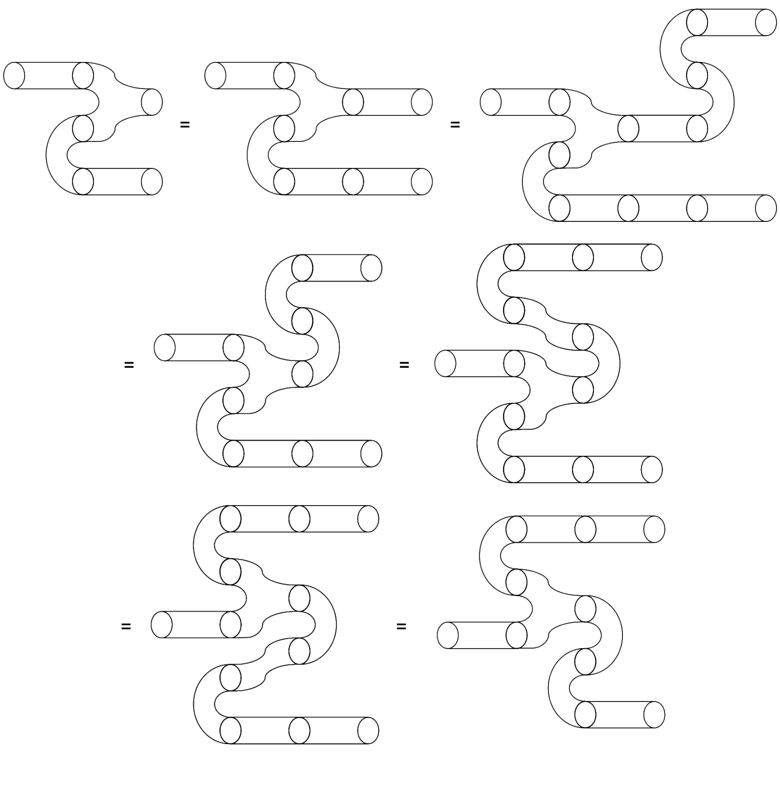
3.2.2 Construction of a comultiplication

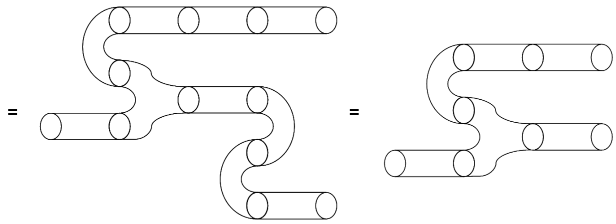
The aim of this section is to construct a comultiplication map on a Frobenius algebra to get a coalgebra structure. Since we already have a map $\varepsilon : A \to k$, namely the Frobenius form, we construct the comultiplication in such a way that the Frobenius form will become the counit.

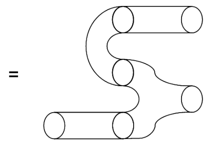
Definition 31. We define a map $\delta : A \to A \otimes A$ by the following picture:



Remark 32. We quickly convince ourselves that the second equality in Definition 31 holds



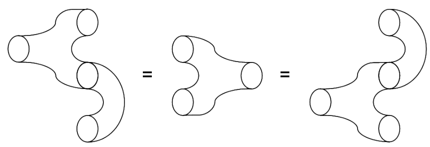




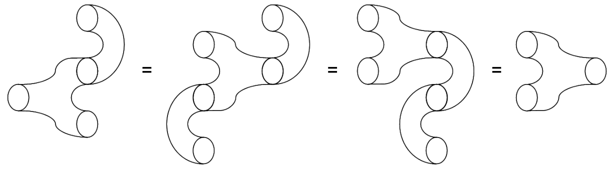
Note that this is not just an array of fancy pictures but a serious mathematical proof since we could easily translate the pictures into the language of ordinary maps and commutative diagrams. The important steps in the proof were to use the snake relation in the first line, then the associativity of the pairing (line skip from line two to line three) and again the snake relation in line four. Other than that we simply inserted some identities here and there which is obviously harmless. Hence, from now on we completely omit identities in our pictures for the sake of brevity.

To see a first proof with omitted identities which is very similar to the detailed proof given above we note the following proposition.

Proposition 33. *The following relations hold:*

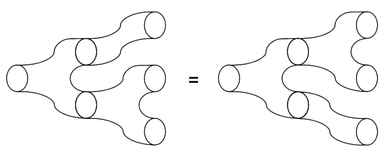


Proof. We will show the right equality. The left one works analogously.

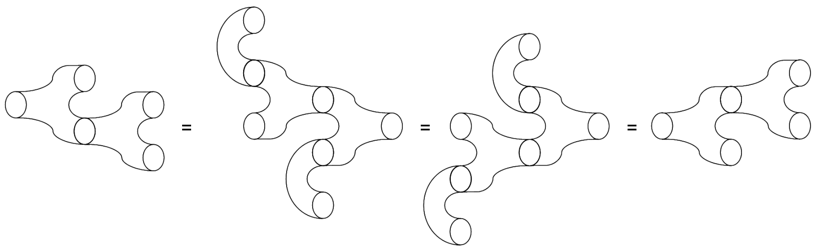


The first equality is just the definition of δ, the second one is the associativity of β and the last one is the snake relation. □

Lemma 34. *The comultiplication δ defined pictorially above is coassociative. In pictures*

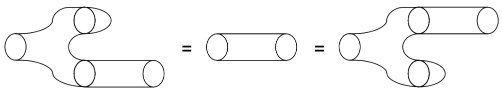


Proof. Consider the pictures

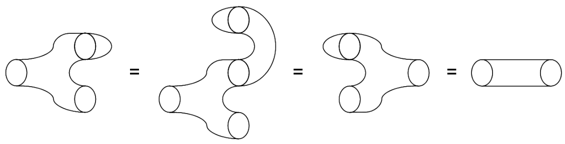


The outer equalities are simply the definition of δ and in the middle we use the associativity of μ. $\square$

Lemma 35. *The Frobenius form ε is the counit for δ*

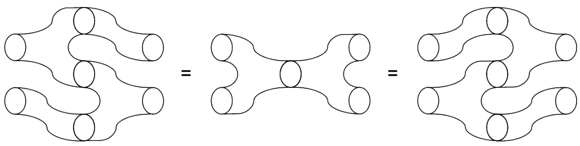


Proof. We only show the left part.



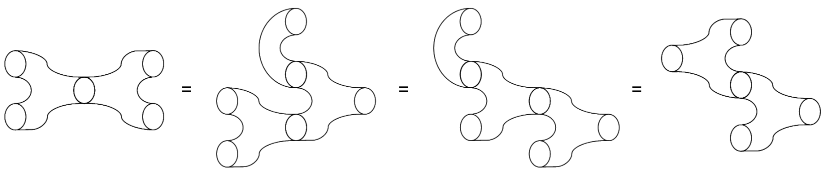
The first equality is the connection between β and ε, the second equality is Proposition 33, and the last one is the unit axiom for a k-algebra. $\square$

Theorem 36. *Let A be a Frobenius Algebra with Frobenius form ε. Then there exists a unique comulitplication whose counit is ε and which satisfies the following relation:*



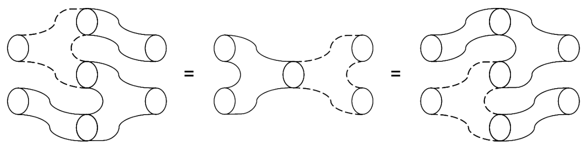
This is called the Frobenius relation.

Proof. In Definition 31 such a comultiplication has been constructed. Lemma 34 establishes its coassociativity and Lemma 35 shows that ε is its counit. To see that the Frobenius relation holds consider the following pictures

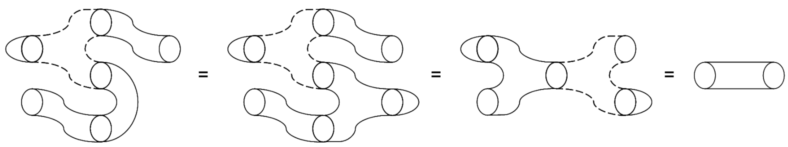


The outer equalities follow from the definition of the comultiplication and in the middle we used the associativity of μ. This shows the left-hand equation of the Frobenius relation. The other one follows analogously.

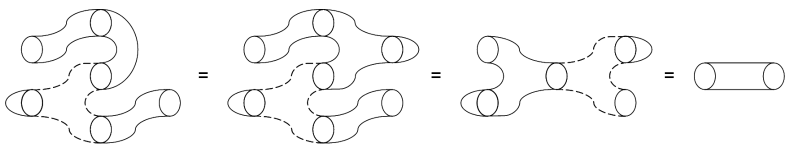
What remains to be shown is the uniqueness. Since we require the comultiplication to satisfy the Frobenius relation we have



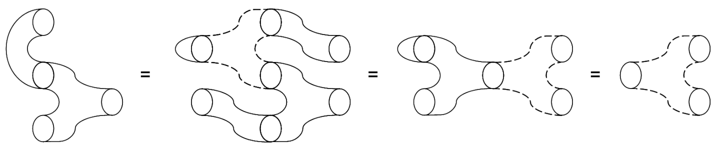
where the dashed symbol stands for an arbitrary comultiplication $\tilde{\delta}$ with the desired properties. From this we obtain



and analogously



by the unit and counit axioms. Hence we have shown that $\tilde{\delta} \circ \eta$ is a copairing for β since it satisfies the snake relation. By Corollary 24 the copairing is unique and thus we have $\gamma = \tilde{\delta} \circ \eta$. This yields



which shows that $\delta = \tilde{\delta}$ since the left side is just the definition of δ (cf. Definiton 31). $\square$

4 Algebraic classification of TQFTs

In this section two different types of TQFTs $F : \mathcal{C} \to \mathbf{Vect}_k$ are studied by specifying a certain monoidal category $\mathcal{C}$.

First we look at 2d-TQFTs $F : \mathbf{2Cob} \to \mathbf{Vect}_k$ in the sense of Atiyah. If F is not only monoidal but also respects the symmetric structure of the two categories involved, such TQFTs correspond to commutative Frobenius algebras. This is the first main result of this thesis. It will allow us to use examples of commutative Frobenius algebras to construct explicit examples of 2d-TQFTs. Moreover, we will see the connection between TQFTs and manifold invariants in the subsequent section.

The second kind of TQFT which is examined afterwards will be certain monoidal functors $F : \mathcal{C} \to \mathbf{Vect}_k$ called *fiber functors* where the domain category will be a *finite k-linear abelian monoidal category.* The main result in this context will be the correspondence of these functors to bialgebras (cf. Theorem 43).

Finally, an attempt is made to contrast the two results. Throughout this section we will work with strictified categories only.

4.1 2d-TQFTs and Frobenius algebras

After the thorough discussion of Frobenius algebras and the category **2Cob** we are ready to prove the following main result straightaway.

Theorem 37. *There is a bijection between*

1. *strict monoidal functors* $F : \mathbf{2Cob} \to \mathbf{Vect}_k$ *which are symmetric, i.e.* $F(\tau_{\mathbf{n},\mathbf{m}}) = \tau_{F(\mathbf{n}),F(\mathbf{m})}$ *for all objects* $\mathbf{n}, \mathbf{m} \in \mathbf{2Cob}$

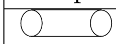
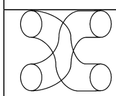
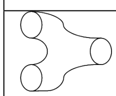
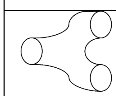
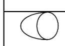
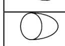
2. *commutative Frobenius algebras.*

Proof. Given a commutative Frobenius algebra A a functor $F : \mathbf{2Cob} \to \mathbf{Vect}_k$ can be defined by setting $F(\mathbf{1}) := A$. Strict monoidality then implies

$$F(\mathbf{n}) = \underbrace{A \otimes ... \otimes A}_{n \text{ times}}.$$

Thus F is completely determined on objects as soon as $F(\mathbf{1})$ is specified.

Recall that Theorem 21 says that any cobordism can be built from the six elementary cobordisms by composition or paralleling. Thus by using functoriality and monoidality again it suffices to specify F for these elementary cobordisms. By Theorem 36 A has a unique structure of a coalgebra such that the Frobenius form is the counit and the Frobenius relation is satisfied. So we define F on morphisms by the following table

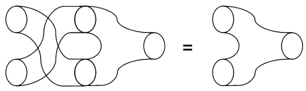
Morphism in **2Cob**	Morphism in $\mathbf{Vect}_k$
	$id : A \to A$
	$\tau : A \otimes A \to A \otimes A$
	$\mu : A \otimes A \to A$
	$\eta : k \to A$
	$\delta : A \to A \otimes A$
	$\varepsilon : A \to k$

This yields a well-defined symmetric monoidal functor since the relations in **2Cob** correspond precisely to the axioms of a commutative Frobenius algebra (cf. Appendix B).[9]

Given a symmetric monoidal functor $F : \mathbf{2Cob} \to \mathbf{Vect}_k$ we can look at $A := F(\mathbf{1})$ which is by definition a finite-dimensional vector space. The idea is to show that A is in fact a Frobenius algebra.

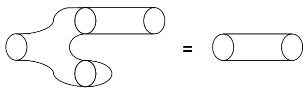
In this case we can use the table above to define maps μ, δ, η and ε as images of the respective cobordism classes.

Notice that μ and η defined in this way satisfy the associativity and unit axiom condition of a k-algebra simply because these relations are true in **2Cob** (cf. Appendix B) and are preserved by the monoidal functor. Since we have



in **2Cob** and F is symmetric this relation passes over to $\mu \circ \tau = \mu$ by applying F. Thus A is a commutative algebra.

For A to be a commutative Frobenius algebra it remains to construct an associative, non-degenerate pairing. This is done by setting $\beta := \varepsilon \circ \mu$. Clearly β is associative because μ is associative. To see the non-degeneracy we define a copairing $\gamma := \delta \circ \eta$. First observe that the Frobenius relation in **2Cob** gives $\mu \otimes id \circ id \otimes \delta = \delta \circ \mu$ by applying F. Moreover we get $\varepsilon \otimes id \circ \delta = id$ since we have



in **2Cob**. We can now use these two equalitites to see

$$\beta \otimes id \circ id \otimes \gamma = \varepsilon \otimes id \circ \underbrace{\mu \otimes id \circ id \otimes \delta}_{=\delta \otimes \mu} \circ id \otimes \eta = \underbrace{\varepsilon \otimes id \circ \delta}_{=id} \circ \underbrace{\mu \circ id \otimes \eta}_{=id} = id$$

where we also used the unit axiom established above for the last equality. Thus we have established the first diagram expressing non-degeneracy. The other one follows analogously. So A is indeed a Frobenius algebra.

The two mappings described above are obviously inverse to each other.

□

4.2 Examples and manifold invariants

In the following two concrete and typical examples of 2d-TQFTs are discussed by specifying a Frobenius algebra. As an application we will look at manifold invariants which the TQFT produces and seize the opportunity of

[9] We have not proven all the equalities listed in Appendix B for Frobenius algebras. Fortunately all the ones left out are straightforward, except for the cocommutativity, where we refer to [Abr97, Thm.2.1.3] or [Koc04, 2.3.29].

doing some explicit calculations. This section was inspired by some of the exercises in [Koc04, pp.176-177].

Example 38 (Nilpotent TQFT)**.** Recall from Example 29 that cohomology rings give rise to Frobenius algebras. To be concrete consider the cohomology ring of $\mathbb{C}P^1$ which is $\mathbb{C}[X]/(X^2)$. This is a $\mathbb{C}$-algebra with basis $\bar{1}, \bar{X}$. The multiplication map μ is then given by

$$\begin{gathered}\bar{1} \otimes \bar{1} \mapsto \bar{1} \\ \bar{X} \otimes \bar{1} \mapsto \bar{X} \\ \bar{1} \otimes \bar{X} \mapsto \bar{X} \\ \bar{X} \otimes \bar{X} \mapsto \bar{0}\end{gathered}$$

and the unit η is simply

$$1 \mapsto \bar{1}.$$

In addition to that we have a Frobenius form $\varepsilon : \mathbb{C}[X]/(X^2) \to \mathbb{C}$ defined by

$$\begin{gathered}\bar{1} \mapsto 0 \\ \bar{X} \mapsto 1.\end{gathered}$$

Using the identity $\beta = \varepsilon \circ \mu$ we immediately calculate that the corresponding pairing β is

$$\begin{gathered}\bar{1} \otimes \bar{1} \mapsto \bar{1} \mapsto 0 \\ \bar{X} \otimes \bar{1} \mapsto \bar{X} \mapsto 1 \\ \bar{1} \otimes \bar{X} \mapsto \bar{X} \mapsto 1 \\ \bar{X} \otimes \bar{X} \mapsto \bar{1} \mapsto 0.\end{gathered}$$

Since it will be important let us calculate the corresponding copairing γ. From the proof of Lemma 23 we know that we can put the images of the basis vectors under β into a matrix as follows

$$\begin{pmatrix}\beta_{11} & \beta_{12} \\ \beta_{21} & \beta_{22}\end{pmatrix} = \begin{pmatrix}\beta(\bar{1} \otimes \bar{1}) & \beta(\bar{1} \otimes \bar{X}) \\ \beta(\bar{X} \otimes \bar{1}) & \beta(\bar{X} \otimes \bar{X})\end{pmatrix} = \begin{pmatrix}0 & 1 \\ 1 & 0\end{pmatrix}$$

and invert this matrix to get

$$\begin{pmatrix}\gamma_{11} & \gamma_{12} \\ \gamma_{21} & \gamma_{22}\end{pmatrix} = \begin{pmatrix}\beta_{11} & \beta_{12} \\ \beta_{21} & \beta_{22}\end{pmatrix}^{-1} = \begin{pmatrix}0 & 1 \\ 1 & 0\end{pmatrix}^{-1} = \begin{pmatrix}0 & 1 \\ 1 & 0\end{pmatrix}$$

where the γ_{ij} are the coefficients of the expansion of the image vector $\gamma(1)$ in the canonical basis. Hence the copairing γ is given by

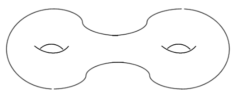
$$1 \mapsto \bar{X} \otimes \bar{1} + \bar{1} \otimes \bar{X}.$$

Last but not least the comuliplication δ can be calculated by looking at $id \otimes \mu \circ \gamma \otimes id$ (cf. Definition 31). So we get

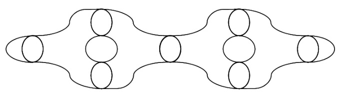
$$\bar{1} \mapsto \bar{X} \otimes \bar{1} + \bar{1} \otimes \bar{X}$$
$$\bar{X} \mapsto \bar{X} \otimes \bar{X}.$$

By the proof of Theorem 37 the commutative Frobenius Algebra $\mathbb{C}[X]/(X^2)$ defines a 2d-TQFT. Since $\mathbb{C}[X]/(X^2)$ is nilpotent the TQFT corresponding to this Frobenius algebra is called nilpotent.

As an application we want to look at manifold invariants produced by this TQFT. So let M be a closed, oriented 2-manifold. The crucial point is that M can always be interpreted as an oriented cobordism $\emptyset \Rightarrow \emptyset$. Thus M determines a certain cobordism class and therefore an arrow in **2Cob**. So the TQFT assigns a k-linear map $k \to k$ to the cobordism class of the manifold M which we simply interpret as an element of k via the canonical identification $k \cong \mathrm{End}(k)$. In particular diffeomorphic manifolds are sent to the same element. Hence we have constructed a diffeomorphism invariant. As an example consider a manifold of genus 2



Its cobordism class can be built from the classes of the basic cobordisms of Theorem 21.



The corresponding linear map $k \to k$ under the TQFT is $\varepsilon \circ \mu \circ \delta \circ \mu \circ \delta \circ \eta$. We observe that $\mu \circ \delta \circ \mu \circ \delta = 0$ by checking that $\mu \circ \delta(\bar{X}) = 0$ and

$$\mu \circ \delta \circ \mu \circ \delta(\bar{1}) = \mu \circ \delta(\bar{X} + \bar{X}) = 0$$

on the basis $\bar{1}$, $\bar{X}$ of A. Thus the assigned map $k \to k$ is the zero map. In particular, one can already see that all manifolds of higher genus will have invariant 0. As a conclusion we see that this TQFT produces stupid invariants. This might be a motivation to look at yet another example.

Example 39 (Semi-simple TQFT)**.** Take the group $\mathbb{Z}/2\mathbb{Z}$ and consider its group algebra $\mathbb{C}[\mathbb{Z}/2\mathbb{Z}]$ over $\mathbb{C}$. We already know that this is an example of a Frobenius algebra. Notice that we have an isomorphism of algebras $\mathbb{C}[\mathbb{Z}/2\mathbb{Z}] \xrightarrow{\sim} \mathbb{C}[X]/(X^2 - 1)$ by sending the canonical basis to the canonical basis. In order to tie in with the notation used in the first example we will describe everything in terms of the algebra $\mathbb{C}[X]/(X^2 - 1)$. By similar calculations as in the example above we obtain the following table

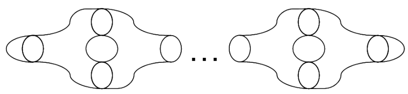
$\mu : A \otimes A \to A$	$\bar{1} \otimes \bar{1} \mapsto \bar{1}$ $\bar{X} \otimes \bar{1} \mapsto \bar{X}$ $\bar{1} \otimes \bar{X} \mapsto \bar{X}$ $\bar{X} \otimes \bar{X} \mapsto \bar{1}$
$\eta : k \to A$	$1 \mapsto \bar{1}$
$\varepsilon : A \to k$	$\bar{1} \mapsto 1$ $\bar{X} \mapsto 0$
$\beta : A \otimes A \to k$	$\bar{1} \otimes \bar{1} \mapsto 1$ $\bar{X} \otimes \bar{1} \mapsto 0$ $\bar{1} \otimes \bar{X} \mapsto 0$ $\bar{X} \otimes \bar{X} \mapsto 1$
$\gamma : k \to A \otimes A$	$1 \mapsto \bar{1} \otimes \bar{1} + \bar{X} \otimes \bar{X}$
$\delta : A \to A \otimes A$	$\bar{1} \mapsto \bar{1} \otimes \bar{1} + \bar{X} \otimes \bar{X}$ $\bar{X} \mapsto \bar{1} \otimes \bar{X} + \bar{X} \otimes \bar{1}$

Again by Theorem 37 this defines a 2d-TQFT. By Maschke's Theorem $\mathbb{C}[X]/(X^2 - 1)$ is semi-simple. This holds in general for any Frobenius algebra over $\mathbb{C}$ obtained from the group algebra of a finite group. That is why the TQFTs obtained from these Frobenius algebras are called semi-simple.

We now want to show that this TQFT can distinguish 2-manifolds of different genus and therefore provides a sensible invariant. For notational convenience we introduce the *handle operator* $h := \mu \circ \delta : A \to A$. By looking at the table above we see that $h(\bar{1}) = \bar{1} + \bar{1} = \bar{2}$. Induction then yields

$$h^k(\bar{1}) = \underbrace{(h \circ ... \circ h)}_{k \text{ times}}(\bar{1}) = h(h^{k-1}(\bar{1})) = h(\underbrace{\bar{1} + ... + \bar{1}}_{2^{k-1} \text{ times}}) = \underbrace{\bar{1} + ... + \bar{1}}_{2^k \text{ times}} = \bar{2}^k.$$

Now consider a two-dimensional manifold of genus k. We cut this manifold as suggested by the following picture



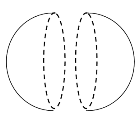
Going over to diffeomorphism classes and applying the TQFT functor we get the linear map

$$\varepsilon \circ \mu \circ \delta \circ ... \circ \mu \circ \delta \circ \eta = \varepsilon \circ h^k \circ \eta$$

which gives the invariant

$$\varepsilon(h^k(\underbrace{\eta(1)}_{=\bar{1}})) = \varepsilon(\underbrace{\bar{1} + ... + \bar{1}}_{2^k \text{ times}}) = 2^k.$$

If we cut the sphere as follows



we get the invariant

$$(\varepsilon \circ \eta)(1) = 1.$$

All in all we have seen that this TQFT assigns the invariant 2^k to a 2-manifold of genus k. This seems like a very strong result. However, we should not be too excited about it because the existence of the classification theorem (cf. Theorem 20), which is always in the background, made the construction of this invariant possible in the first place. So in fact we have not gained anything. Nonetheless we can already see that higher-dimensional TQFTs might be promising theories to classify higher-dimensional manifolds where such complete classification results do not exist.

4.3 TQFTs and bialgebras

In this section we replace the category **2Cob** and study monoidal functors/TQFTs whose domain category is a *finite k-linear abelian monoidal category*. Instead of defining all these words, we will use a characterization of these categories which says that a *finite k-linear abelian monoidal category* is equivalent to a category $A-\mathbf{mod}$ of finite-dimensional modules over a finite-dimensional k-algebra A, see [EGNO, p.40] and [Fre64, Chapter 7] for more on this. For a definition which does not use this equivalence and instead explains each word independently, see [CE08].[10] In the following $\mathcal{C}$ always stands for a *finite k-linear abelian monoidal category*. The following is a worked out version of [EGNO, pp.40-43].

Lemma 40. *Let $F : \mathcal{C} \to \mathbf{Vect}_k$ be a functor. Then the collection* $\mathrm{End}(F)$ *of all natural transformations $\mu : F \to F$ can be equipped with the structure of a k-algebra.*

Proof. Let η and μ denote natural transformations from F to itself. Then the sum $\eta + \mu$ is given by $(\eta + \mu)_X := \eta_X + \mu_X$ where $\eta_X + \mu_X$ denotes the morphism $F(X) \to F(X)$ defined by $v \mapsto \eta_X(v) + \mu_X(v)$. Notice that $\eta + \mu$ is indeed a well-defined natural transformation since for every morphism $f : X \to Y$ in $\mathcal{C}$ we have

$$\begin{aligned} F(f)((\eta + \mu)_X(v)) &= F(f)(\eta_X(v) + \mu_X(v)) \\ &= F(f)(\eta_X(v)) + F(f)(\mu_X(v)) \\ &= \eta_Y(F(f)(v)) + \mu_Y(F(f)(v)) \\ &= (\eta + \mu)_Y(F(f)(v)). \end{aligned}$$

Analogously we define $(\lambda\eta)_X := \lambda\eta_X$ with $\lambda\eta_X : F(X) \to F(X)$, $v \mapsto \lambda\eta_X(v)$ and $(\eta \cdot \mu)_X := \eta_X \circ \mu_X$ and check that we get natural transformations. Since the endomorphisms of a vector space constitute an algebra it

[10] This might in fact be a better approach because the category $A - \mathbf{mod}$ corresponding to a finite k-linear abelian monoidal category is not unique. It is unique only up to the Morita equivalence class of A.

is clear that these operations turn $\mathrm{End}(F)$ into a k-algebra where the zero element is given by the transformation consisting of zero maps only and the unit element is given by the transformation consisting of identities in each component. $\square$

Given a functor $F : \mathcal{C} \to \mathbf{Vect}_k$ we can define a functor $F \otimes F : \mathcal{C} \times \mathcal{C} \to \mathbf{Vect}_k$ by $(F \otimes F)(X, Y) := F(X) \otimes F(Y)$ by using the tensor product in $\mathbf{Vect}_k$. Now it makes sense to consider the algebra $\mathrm{End}(F \otimes F)$. This algebra is easy to understand in terms of the algebra $\mathrm{End}(F)$ since we have the following result

Lemma 41. *There is an isomorphism of k-algebras $\alpha_F : \mathrm{End}(F) \otimes \mathrm{End}(F) \to \mathrm{End}(F \otimes F)$ given by*

$$\alpha_F(\eta \otimes \mu)_{X,Y} := \eta_X \otimes \mu_Y$$

where $\eta, \mu \in \mathrm{End}(F)$.

Proof. The first observation is that α_F is a well-defined homomorphism of k-algebras. Consider the map $\tilde{\alpha}_F : \mathrm{End}(F) \times \mathrm{End}(F) \to \mathrm{End}(F \otimes F)$ given by

$$\tilde{\alpha}_F(\eta, \mu)_{X,Y} := \eta_X \otimes \mu_Y.$$

This is a k-bilinear map. For the first component the equation $\tilde{\alpha}_F(\eta+\tilde{\eta}, \mu) = \tilde{\alpha}_F(\eta, \mu) + \tilde{\alpha}_F(\tilde{\eta}, \mu)$ follows from

$$\begin{aligned} \tilde{\alpha}_F(\eta + \tilde{\eta}, \mu)_{X,Y} &= (\eta + \tilde{\eta})_X \otimes \mu_Y \\ &= (\eta_X + \tilde{\eta}_X) \otimes \mu_Y \\ &= \eta_X \otimes \mu_Y + \tilde{\eta}_X \otimes \mu_Y \\ &= \tilde{\alpha}_F(\eta, \mu)_{X,Y} + \tilde{\alpha}_F(\tilde{\eta}, \mu)_{X,Y} \\ &= (\tilde{\alpha}_F(\eta, \mu) + \tilde{\alpha}_F(\tilde{\eta}, \mu))_{X,Y}. \end{aligned}$$

To see that $\tilde{\alpha}_F(\lambda.\eta, \mu) = \lambda.\tilde{\alpha}_F(\eta, \mu)$ we calculate

$$\begin{aligned} \tilde{\alpha}_F(\lambda.\eta, \mu)_{X,Y} &= (\lambda.\eta)_X \otimes \mu_Y \\ &= (\lambda.\eta_X) \otimes \mu_Y \\ &= \lambda.(\eta_X \otimes \mu_Y) \\ &= \lambda.\tilde{\alpha}_F(\eta, \mu)_{X,Y}. \end{aligned}$$

Similar calculations can be done for the other component. Thus we see that $\tilde{\alpha}_F$ induces the k-linear map α_F by the universal property of the tensor product. In addition to that we have $\alpha_F(\eta \otimes \mu \cdot \tilde{\eta} \otimes \tilde{\mu}) = \alpha_F(\eta \otimes \mu) \cdot \alpha_F(\tilde{\eta} \otimes \tilde{\mu})$

because

$$\begin{aligned}
\alpha_F(\eta\otimes\mu\cdot\tilde{\eta}\otimes\tilde{\mu})_{X,Y} &= \alpha_F(\eta\cdot\tilde{\eta}\circ\mu\cdot\tilde{\mu})_{X,Y}\\
&= (\eta\cdot\tilde{\eta})_X\otimes(\mu\cdot\tilde{\mu})_Y\\
&= (\eta_X\circ\tilde{\eta}_X)\otimes(\mu_Y\circ\tilde{\mu}_Y)\\
&= (\eta_X\otimes\mu_Y)\circ(\tilde{\eta}_X\otimes\tilde{\mu}_Y)\\
&= \alpha_F(\eta\otimes\mu)_{X,Y}\circ\alpha_F(\tilde{\eta}\otimes\tilde{\mu})_{X,Y}.
\end{aligned}$$

All in all we have shown that α_F is a k-algebra homomorphism which is obviously unital.

It remains to make sure that α_F is a bijection. It suffices to show that

$$\mathrm{End}(F(X))\otimes\mathrm{End}(F(Y))\to\mathrm{End}(F(X)\otimes F(Y))$$

given by

$$(f:F(X)\to F(X))\otimes(g:F(Y)\to F(Y))\mapsto(f\otimes g:F(X)\otimes F(Y)\to F(X)\otimes F(Y))$$

is a bijection for any particular choice of $X,Y\in\mathcal{C}$. Then we see that the homomorphism $\alpha_F:\mathrm{End}(F)\otimes\mathrm{End}(F)\to\mathrm{End}(F\otimes F)$ given by

$$\alpha_F(\eta\otimes\mu)_{(X,Y)}:=\eta_X\otimes\mu_Y$$

is bijective as well, simply by applying the bijection above in each component. The proof of this bijection is standard, see e.g. [Kas95, Thm.II2.1]. □

From now on let $F:\mathcal{C}\to\mathbf{Vect}_k$ be an exact and faithful monoidal functor such that $\phi:F(\mathbf{1})\xrightarrow{\sim}k$ is the identity (cf. Definition 10). Such a functor is called a *fiber functor*.

Theorem 42. *Let $F:\mathcal{C}\to\mathbf{Vect}_k$ be a fiber functor. Then the k-algebra* $\mathrm{End}(F)$ *can be equipped with a comultiplication δ and a counit ε which turn* $\mathrm{End}(F)$ *into a bialgebra.*[11]

Proof. In order to avoid confusion with the multiplication and unit in an algebra we will denote natural transformations in $\mathrm{End}(F)$ by small Roman letters. Now define $\delta:\mathrm{End}(F)\to\mathrm{End}(F)\otimes\mathrm{End}(F)$ to be

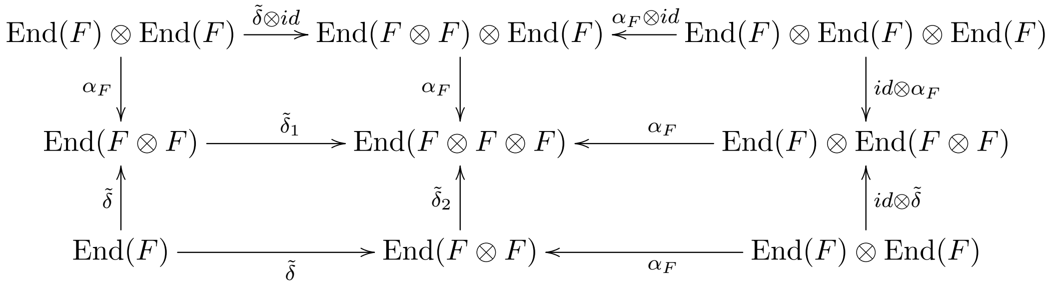
$$\delta(a):=\alpha_F^{-1}(\tilde{\delta}(a))$$

where $\tilde{\delta}(a)\in\mathrm{End}(F\otimes F)$ is given by

$$\tilde{\delta}(a)_{X,Y}:=J_{X,Y}^{-1}a_{X\otimes Y}J_{X,Y}.$$

[11]Even though the functor is monoidal we do not require the transformations in $\mathrm{End}(F)$ to be natural monoidal transformations.

This is a k-linear map because α_F^{-1} is k-linear and the linearity of $\tilde{\delta}$ is clear. To see the coassociativity of this comultiplication consider the following diagram



where $\tilde{\delta}_1$ applies $\tilde{\delta}$ to the first factor and leaves the second one unchanged and similarly for $\tilde{\delta}_2$. Notice that by the definition of δ the commutativity of the outer square is equivalent to the coassociativity. Thus it suffices to show the commutativity of the four small squares. The only square which is interesting is in fact the one down left. Checking the commutativity of the other ones is elementary.

So take $a \in \mathrm{End}(F)$. Since $\tilde{\delta}(a)_{X,Y} = J_{X,Y}^{-1} a_{X \otimes Y} J_{X,Y}$ we have

$$\tilde{\delta}_1(\tilde{\delta}(a))_{X,Y,Z} = J_{X,Y}^{-1} \otimes id_{F(Z)} \circ J_{X \otimes Y,Z}^{-1} a_{X \otimes Y \otimes Z} J_{X \otimes Y,Z} \circ J_{X,Y} \otimes id_{F(Z)}$$

for chosen objects X, Y, Z by the definition of $\tilde{\delta}_1$. Similarly we have

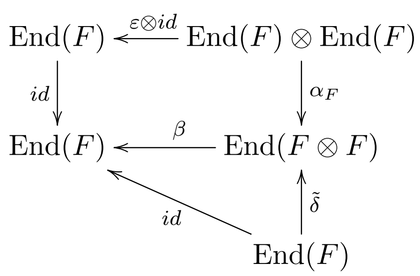
$$\tilde{\delta}_2(\tilde{\delta}(a))_{X,Y,Z} = id_{F(X)} \otimes J_{Y,Z}^{-1} \circ J_{X,Y \otimes Z}^{-1} a_{X \otimes Y \otimes Z} J_{X,Y \otimes Z} \circ id_{F(X)} \otimes J_{Y,Z}.$$

But these maps are equal since we have

$$J_{X \otimes Y,Z} \circ J_{X,Y} \otimes id_{F(Z)} = J_{X,Y \otimes Z} \circ id_{F(X)} \otimes J_{Y,Z}$$

by the monoidal structure axiom (notice that the associativity constraints are gone since our categories are strict).

Now define a counit $\varepsilon : \mathrm{End}(F) \to k$ by setting $\varepsilon(a) := a_1$. To actually obtain an element in k we identify a_1 with $a_1(1)$. Consider the diagram



with $\beta : \mathrm{End}(F \otimes F) \to \mathrm{End}(F)$ given by

$$\beta(\eta)_X : F(X) = F(1) \otimes F(X) \xrightarrow{\eta_{1,X}} F(1) \otimes F(X) = F(X)$$

where the equalities are the strictified unit constraints (remember that $F(1) = k$). If this diagram commutes we obtain that ε is a left counit. So let us investigate the small diagrams starting with the upper square.

Let $a \otimes b \in \mathrm{End}(F) \otimes \mathrm{End}(F)$. Then we have $\varepsilon \otimes id(a \otimes b) = a_1(1).b \in \mathrm{End}(F)$ on the one side. Chasing through the square via $\beta \circ \alpha_F$ we obtain

$$\beta(\alpha_F(a \otimes b))_X : F(X) = F(1) \otimes F(X) \xrightarrow{a_1 \otimes b_X} F(1) \otimes F(X) = F(X),$$

thus explicitly on elements

$$\beta(\alpha_F(a \otimes b))_X(v) = a_1(1).b_X(v)$$

which is exactly the transformation $a_1(1).b$. For the triangle notice that for $a \in \mathrm{End}(F)$ we have

$$\beta(\tilde{\delta}(a))_X : F(X) = F(1) \otimes F(X) \xrightarrow{J^{-1}_{\mathbf{1},X} a_{\mathbf{1},X} J_{\mathbf{1},X}} F(1) \otimes F(X) = F(X)$$

which collapses to $F(1) \otimes F(X) \xrightarrow{a_{\mathbf{1},X}} F(1) \otimes F(X)$ since $J_{\mathbf{1},X} = id$ by the second diagram in Definition 10. Finally, $a_{\mathbf{1},X}$ is identified with a_X by strictness. The proof that ε is also a right unit is analogous.
Furthermore δ is an algebra homomorphism. We have

$$\tilde{\delta}(a)_{X,Y}\tilde{\delta}(b)_{X,Y} = J^{-1}_{X,Y} a_{X,Y} b_{X,Y} J_{X,Y} = J^{-1}_{X,Y}(ab)_{X,Y} J_{X,Y} = \tilde{\delta}(ab)_{X,Y}$$

by the definition of $\tilde{\delta}$ and the definition of the multiplication in $\mathrm{End}(F \otimes F)$. Hence we get

$$\delta(a)\delta(b) = \alpha_F^{-1}(\tilde{\delta}(a))\alpha_F^{-1}(\tilde{\delta}(b)) = \alpha_F^{-1}(\tilde{\delta}(a)\tilde{\delta}(b)) = \alpha_F^{-1}(\tilde{\delta}(ab)) = \delta(ab)$$

because α_F is an isomorphism of algebras. It is obvious that δ is unital. Moreover, ε is clearly a unital algebra homomorphism. All in all we have proven that $\mathrm{End}(F)$ has the structure of a bialgebra. □

Theorem 43. *There is a bijection between*

1. *finite k-linear abelian monoidal categories $\mathcal{C}$ together with a fiber functor $F : \mathcal{C} \to \mathbf{Vect}_k$ (up to monoidal equivalence and isomorphism of monoidal functors)*

2. *finite-dimensional bialgebras H over k (up to isomorphism).*

Proof. The bijection goes as follows: Given a fiber functor $F : \mathcal{C} \to \mathbf{Vect}_k$ we assign to it the bialgebra $\mathrm{End}(F)$ of functorial endomorphisms constructed in Theorem 42. On the other hand, given a finite-dimensional bialgebra H we can consider the category $\mathbf{Rep}(H)$ of finite-dimensional modules over H discussed in Example 5. This is a k-linear abelian monoidal category. The functor $F : \mathbf{Rep}(\mathrm{H}) \to \mathbf{Vect}_k$ is defined to be the forgetful functor which is obviously a fiber functor.

We quickly sketch why these assignments are mutually inverse. Let us start with a finite k-linear abelian monoidal category together with a fiber functor $F : \mathcal{C} \to \mathbf{Vect}_k$. Since F is by definition exact and faithful it is a well-known result that there exists a unique (up to unique isomorphism) projective generator[12] P of $\mathcal{C}$ such that $F = F_P$ where $F_P : \mathcal{C} \to \mathbf{Vect}_k$ denotes the functor given by $F_P(X) = \mathrm{Hom}(P, X)$. By the characterization of finite k-linear abelian monoidal categories as finite-dimensional module-categories (see p.35 and the references given there), one obtains that $\mathcal{C}$ is monoidally equivalent to the category of $\mathrm{End}(P)^{op}$-modules. But from the above we see that this is nothing but $\mathrm{End}(F_P)$-modules. Thus we obtain that $\mathcal{C}$ is in fact monoidally equivalent to the category $\mathbf{Rep}(\mathrm{End}(F))$ of $\mathrm{End}(F)$-modules. Moreover, composing this equivalence with the forgetful functor equals F.

Starting with a bialgebra H it needs to be verified that $H \cong \mathrm{End}(F)$ as bialgebras where $F : \mathbf{Rep}(H) \to \mathbf{Vect}_k$ denotes the forgetful functor. It is straightforward from Example 5 and the proof of Theorem 42 that the map $H \to \mathrm{End}(F)$ sending h to the transformation η_h given by

$$(\eta_h)_{(V,\phi)} := \phi(h) \in \mathrm{End}_k(V),$$

where (V, ϕ) denotes some representation, is an isomorphism of bialgebras. □

4.4 Comparison of the main results

After having established Theorem 37 and Theorem 43 it is natural to ask whether these results are mathematically connected on a deeper level in addition to the similarity regarding their formulation, i.e. both describe a bijection between some kind of a TQFT and a distinguished algebraic structure. A first attempt to connect these results could be undertaken by understanding the connection between bialgebras and Frobenius algebras. By Theorem 36 we know that a Frobenius algebra has a unique structure of a coalgebra such that the Frobenius form is the counit. Thus one might be tempted to hope that Frobenius algebras turn out to be bialgebras via

[12] The condition that we have enough projectives and only a finite number of isomorphism classes of simple objects is part of the definition of being a finite category (see [EGNO, 1.18.2]). Take a projective cover of a simple object from each isomorphism class. Then their direct sum constitutes a projective generator of $\mathcal{C}$.

this construction. This is wrong because in general there is no reason why the Frobenius form and the comultiplication should be homomorphisms of algebras. Vice versa in most cases bialgebras are not Frobenius algebras. These thoughts are made precise by the following theorem.

Theorem 44. *Let A together with maps η, μ, δ and ε (as in Theorem 36) be a Frobenius algebra. The data $(A; \mu, \eta, \delta, \varepsilon)$ defines a bialgebra if and only if A is isomorphic (as Frobenius algebra)*[13] *to the trivial Frobenius algebra k with Frobenius form $\varepsilon' = id$ (cf. Example 27).*

Proof. Let A together with η, μ, δ and ε be a bialgebra. Since by definition the counit (which is also the Frobenius form) ε is required to be a homomorphism of algebras it is in particular a homomorphism of rings. So $\ker(\varepsilon) \subset A$ is an ideal. Let $b \in \ker(\varepsilon)$. For an arbitrary $a \in A$ this implies $\mu(a \otimes b) \in \ker(\varepsilon)$ and therefore $\varepsilon(\mu(a \otimes b)) = 0$. Since a was chosen to be arbitrary we have $b = 0$ since ε is a Frobenius form. Thus $\ker(\varepsilon) = 0$ and hence $A \cong A/\ker(\varepsilon) \cong k$ as k-algebras by the homomorphism theorem because ε is surjective ($\varepsilon(1) = 1$ and ε is k-linear).

Notice that the constructed isomorphism $A \xrightarrow{\sim} k$ is given by ε itself. In particular it is an isomorphism of Frobenius algebras from A with ε to k with Frobenius form id.

On the other hand if we begin with the trivial Frobenius algebra k with Frobenius form id we calculate that $\delta : k \to k \otimes k$ is given by $1 \mapsto 1 \otimes 1$ (cf. Section 4.2 for examples of such calculations). Now it is easy to see that ε and δ are in fact homomorphisms of k-algebras. Thus we have a bialgebra whose counit is the Frobenius form.

Moreover, if we start with a Frobenius algebra A with Frobenius form ε which is isomorphic to k together with id it follows from the compatibility condition of the Frobenius forms that $\varepsilon = f$ where f denotes the isomorphism of k-algebras between A and k. Thus it suffices to check that also the comultiplication δ associated with A is a homomorphism of algebras. Since ε is the counit of δ in A we have $\varepsilon \otimes id \circ \delta = id$ by the counit axiom. Thus we obtain

$$\delta = f^{-1} \otimes id \circ \underbrace{f \otimes id \circ \delta}_{=\varepsilon \otimes id \circ \delta = id} = f^{-1} \otimes id \circ id.$$

So we see that δ is an algebra homomorphism as a combination of the homomorphisms f^{-1} and id. $\square$

By this theorem bialgebras and Frobenius algebras are connected only by the trivial example. Thus it seems that Theorem 37 and Theorem 43 are rather different since the obvious connection via the respective algebraic

[13] An isomorphism $f : A \to A'$ of Frobenius algebras is an isomorphism of k-algebras which is compatible with the Frobenius forms, i.e. $\varepsilon' \circ f = \varepsilon$ where ε is the form of A and ε' is the form of A'.

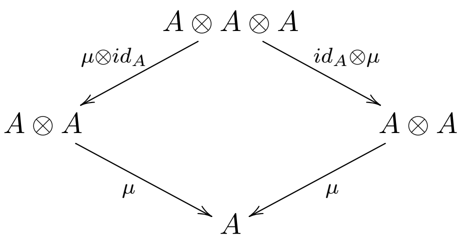
structures fails. It requires further investigation to see whether Theorem 37 could possibly be related in an interesting way to an existing reconstruction theorem of Hopf algebras similar to Theorem 43, see [EGNO, Thm.1.22.11]. The hope to see some interesting connection in this case are based on a theorem asserting that finite-dimensional Hopf algebras can be equipped with the structure of a Frobenius algebra. To see this, one studies the so-called *space of integrals* for a given Hopf algebra by making extensive use of the theory of rational modules. A detailed discussion of a proof would go beyond the scope of this thesis. Instead we refer to [Swe69, Chapter V] for a thorough treatment.

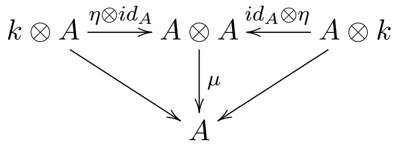
A Algebras, coalgebras and bialgebras

Definition 45. A k-*algebra* is a k-vector space A together with two k-linear maps

$$\mu : A \otimes A \to A \qquad \eta : k \to A$$

such that the following diagrams commute





The map μ is called *multiplication* and η is called *unit*. The diagram on the left expresses the *associativity* of μ and the diagram on the right is referred to as *unit axiom*.

Remark 46. Usually a k-algebra is defined to be a k-vector space together with a bilinear multiplication map $\cdot : A \times A \to A$. Note that by the universal property of the tensor product we have a bijection

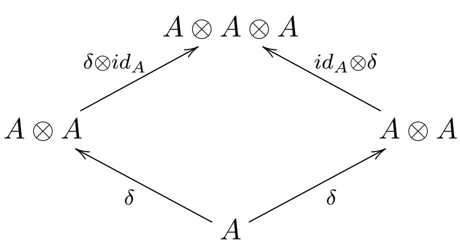
$$\{k\text{-bilinear maps } \cdot : A \times A \to A\} \xrightarrow{1:1} \{k\text{-linear maps } \mu : A \otimes A \to A\}.$$

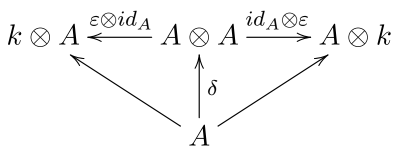
This allows us to switch between both possible definitions whenever it seems convenient.

Definition 47. A *coalgebra* over k is a k-vector space A together with two k-linear maps

$$\delta : A \to A \otimes A \qquad \varepsilon : A \to k$$

such that the following diagrams commute





The map δ is called *comultiplication* and ε is called *counit*. The diagram on the left expresses the *coassociativity* of δ and the diagram on the right is referred to as *counit axiom*.

Definition 48. A *bialgebra* over k is a k-algebra which is also a coalgebra over k and the maps δ and ε are morphisms of algebras, i.e. we have

$$\mu' \circ \delta \otimes \delta = \delta \circ \mu \qquad \text{and} \qquad \delta \circ \eta = \eta'$$

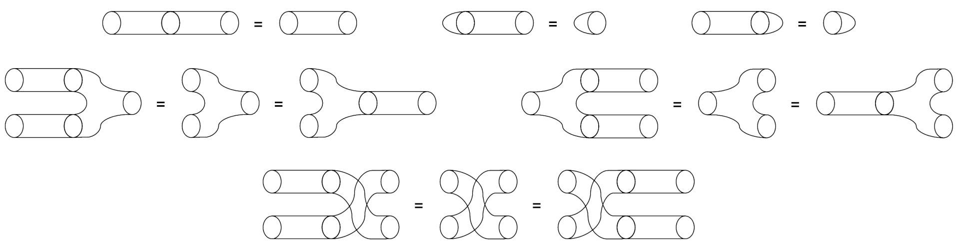
as well as

$$\mu'' \circ \varepsilon \otimes \varepsilon = \varepsilon \circ \mu \qquad \text{and} \qquad \varepsilon \circ \eta = \eta'',$$

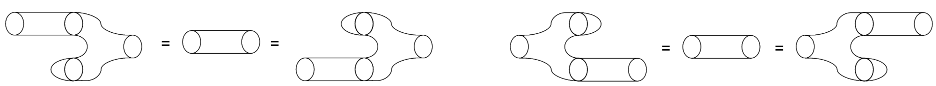
where μ' and η' denote the multiplication and counit in $A \otimes A$, and μ'' and η'' denote the multiplication and counit in k.

B Relations in 2Cob

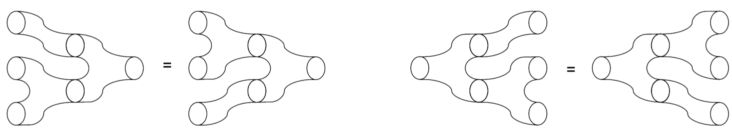
Identity relations



Sewing in discs

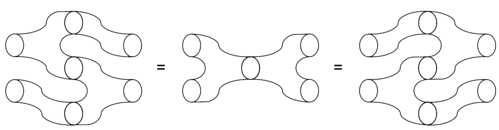


Associativity and coassociativity

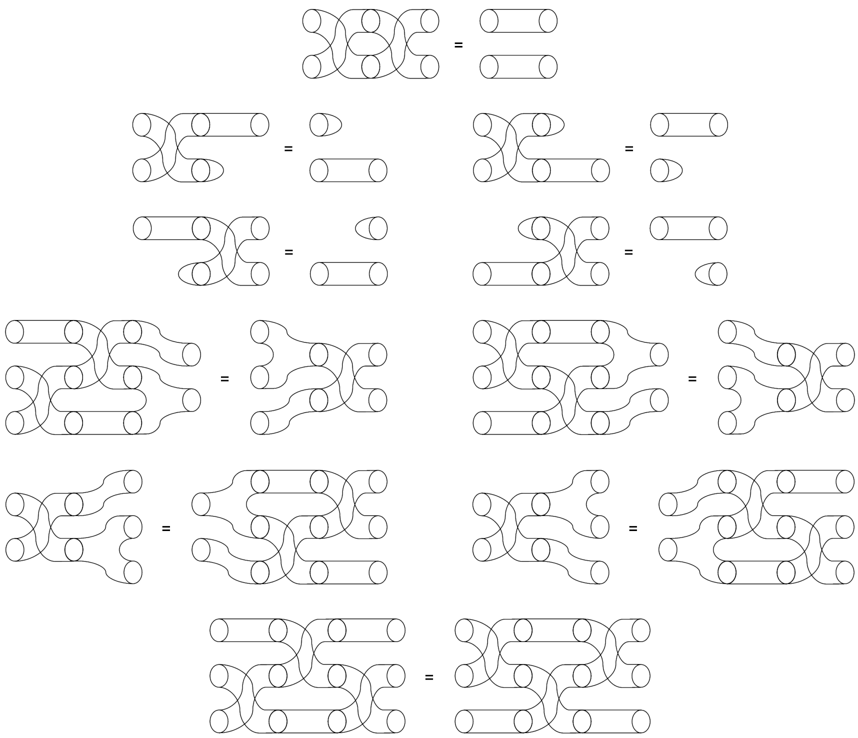


Commutativity and cocommutativity

Frobenius relation



Twist relations



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