Adem relations

Recall: $Sq^i : H^*(X, \mathbb{Z}/2) \longrightarrow H^*(X, \mathbb{Z}/2)$

s.th. $Sq^i(H^n(X,\mathbb{Z}/2)) \subseteq H^{n+i}(X,\mathbb{Z}/2)$

"bar β bracket"
greatest integer $\leq \frac{a}{2}$

Thm: $\underbrace{Sq^a Sq^b}_{\text{composition}} = \sum_{c=0}^{\lfloor \frac{a}{2} \rfloor} \underbrace{\binom{b-c-1}{a-2c}}_{\text{mod } 2} \underbrace{Sq^{a+b-c} Sq^c}_{\text{composition}}, \quad a < 2b$

This is to be understood as equality of maps!

Note: $\binom{p}{q} \equiv 0$ if $q<0$, $\binom{p}{q} = 0$ if $p<q$

Ex: $Sq^3 Sq^2 = \underbrace{\binom{1}{3}}_{=0} Sq^5 Sq^0 + \underbrace{\binom{0}{1}}_{=0} Sq^4 Sq^1 = 0.$

$Sq^1 Sq^2 = \underbrace{\binom{1}{1}}_{=1} Sq^3 \underbrace{Sq^0}_{=id} = Sq^3.$

$Sq^2 Sq^3 = \underbrace{\binom{2}{2}}_{=1} Sq^5 \underbrace{Sq^0}_{=id} + \underbrace{\binom{1}{0}}_{=1} Sq^4 Sq^1 = Sq^5 + Sq^4 Sq^1.$

$Sq^3 Sq^3 = \underbrace{\binom{2}{3}}_{=0} Sq^6 Sq^0 + \underbrace{\binom{1}{1}}_{=1} Sq^5 Sq^1 = Sq^5 Sq^1.$

Need to show:

$$0 = \underbrace{Sq^a Sq^b(y) + \sum_c \binom{b-c-1}{a-2c} Sq^{a+b-c} Sq^c(y)}_{=: R_{a,b}(y), \quad a<2b} \qquad \forall y \in H^*(X, \mathbb{Z}/2) \text{ homogeneous}$$

Will use:

(1) (Serre's thm) Proof later in this seminar!

$\exists f: \underbrace{\mathbb{RP}^\infty \times \dots \times \mathbb{RP}^\infty}_{n\text{-times}} \longrightarrow K(\mathbb{Z}/2, n)$ s.th. ($=: K_n$)

(i) $f^*: H^i(K(\mathbb{Z}/2,n), \mathbb{Z}/2) \longrightarrow H^i(K_n, \mathbb{Z}/2)$ inj. $\forall i \leq 2n$

(ii) $f^*(\iota_n) = \underbrace{x_1 \cdot \ldots \cdot x_n}_{=: \sigma_n}$ (Recall: $H^*(K_n, \mathbb{Z}/2) \cong \mathbb{Z}/2[x_1, \dots, x_n]$)

(2) Lemma: $y \in H^*(X, \mathbb{Z}/2)$ s.th. $R_{a,b}(y) = 0 \ \forall a<2b \Rightarrow R_{a,b}(xy) = 0 \ \forall x \in H^1(X, \mathbb{Z}/2) \ \forall a<2b$

Proof later today!

Prop 1. $y \in H^n(\underset{\text{arbitrary space}}{X}, \mathbb{Z}/2)$, $a+b \leq n \Rightarrow R_{a,b}(y) = 0$ "Adem relations for el. of large degree"

Proof.
- Have: $\underset{H^*(K_n,\mathbb{Z}/2)}{R_{a,b}(\sigma_n)} = 0$

  because: $R_{a,b}(1) = 0$ (by Steenrod prop. 1)

  $R_{a,b}(x_1) = R_{a,b}(x_1 \cdot 1) \underset{\text{lemma}}{=} 0$

  $R_{a,b}(x_2 x_1) = 0$

  $\vdots$ inductively

  $R_{a,b}(x_1 \dots x_n) = 0$

- $f^*(\underbrace{R_{a,b}(i_n)}_{\in H^{n+a+b}(\underbrace{K_n}_{K(\mathbb{Z}/2,n)}, \mathbb{Z}/2)}) \overset{\text{naturality}}{=} R_{a,b}(f^*(i_n)) \overset{\text{Serre}}{=} R_{a,b}(\sigma_n) \overset{\text{see above}}{=} 0 \overset{f^* \text{ inj. (assumption + Serre)}}{\Longrightarrow} R_{a,b}(i_n) = 0$

- **Recall:** $[X, K(\mathbb{Z}/2,n)] \xrightarrow{\cong} H^n(X,\mathbb{Z}/2)$ bij. "Brown representability"

  $[g] \longmapsto g^*(i_n)$

  Pick $g: X \to K(\mathbb{Z}/2,n)$ s.th. $g^*(i_n) = y$.

  $\Rightarrow R_{a,b}(y) = R_{a,b}(g^*(i_n)) = g^*(\underbrace{R_{a,b}(i_n)}_{=0}) = 0$ $\boxed{//}$

Prop 2. $R_{a,b}(y) = 0 \ \forall \tilde{X}, y \in H^p(\tilde{X}, \mathbb{Z}/2) \Rightarrow R_{a,b}(z) = 0 \ \forall X, z \in H^{p-1}(X, \mathbb{Z}/2)$

Proof. $0 \neq u \in H^1(S^1, \mathbb{Z}/2) \cong \mathbb{Z}/2$, $u \times z \in H^p(S^1 \times X, \mathbb{Z}/2)$

Recall: $Sq^i(x \times y) = \sum_{s=0}^{i} Sq^s(x) \times Sq^{i-s}(y)$ "Cartan formula"

$Sq^i(u \times z) \underset{\substack{\text{Cartan formula} \\ + Sq^i(u)=0 \ \forall i>0}}{=} Sq^0(u) \times Sq^i(z) = u \times Sq^i(z)$

$0 \underset{\text{assumption}}{=} R_{a,b}(u \times z) = \underbrace{Sq^a \underbrace{Sq^b(u \times z)}_{u \times Sq^b(z)}}_{u \times Sq^a Sq^b(z)} + \underbrace{\sum_c \binom{b-c-1}{a-2c} Sq^{a+b-c} \underbrace{Sq^c(u \times z)}_{u \times Sq^c(z)}}_{u \times Sq^{a+b-c} Sq^c(z)}$

$= \underset{0}{u} \times R_{a,b}(z) \underset{\mathbb{Z}/2 \text{ is a field}}{\Longrightarrow} R_{a,b}(z) = 0.$ $\boxed{//}$

Note: $\underbrace{\text{Prop 1}}_{\text{base of induction}} + \underbrace{\text{Prop 2}}_{\text{inductive step}} \Longrightarrow$ Thm downward induction.

Proof of lemma:

- (*) $$\underset{\text{Cartan}}{Sq^b(xy) =} \sum_{i=0}^{b} \underbrace{Sq^{b-i}(x)}_{\text{1-dim. class}} Sq^i(y) \underset{\substack{Sq^0(x)=x \\ Sq^1(x)=x^2 \\ Sq^i(x)=0,\ i>1}}{=} x\,Sq^b(y) + x^2 Sq^{b-1}(y)$$

- $$Sq^a Sq^b(xy) \overset{(*)}{=} Sq^a\left(x\,Sq^b(y) + x^2 Sq^{b-1}(y)\right)$$
$$= \underbrace{Sq^a\left(x\,Sq^b(y)\right)}_{\overset{(*)}{=}\ x\,Sq^a Sq^b(y) + x^2 Sq^{a-1} Sq^b(y)} + \underbrace{Sq^a\left(x^2 Sq^{b-1}(y)\right)}_{\underset{\text{Cartan}}{=} \sum_{i=0}^{a} Sq^i(x^2)\, Sq^{a-i} Sq^{b-1}(y)}$$
$$= x^2 Sq^a Sq^{b-1}(y) + x^4 Sq^{a-2} Sq^{b-1}(y)$$

Δ

$Sq^0(x^2)=x^2$

$Sq^2(x^2)=x^4$

$$Sq^1(x^2) \overset{(*)}{=} \underbrace{x\,Sq^1(x)}_{=x^2} + x^2 \underbrace{Sq^0(x)}_{=x} = 2x^3 \underset{\text{mod } 2}{=} 0.$$

- $$\sum_{c} \underbrace{\binom{b-c-1}{a-2c}}_{=:\,s(c)} Sq^{a+b-c} Sq^c(xy) = x \sum_{c} s(c)\, Sq^{a+b-c} Sq^c(y)$$
$$+ x^2 \sum_{c} s(c)\, Sq^{a+b-c-1} Sq^c(y)$$
$$+ x^2 \sum_{c} s(c)\, Sq^{a+b-c} Sq^{c-1}(y)$$
$$+ x^4 \sum_{c} s(c)\, Sq^{a+b-c-2} Sq^{c-1}(y)$$

$\Delta\Delta$

To show: $\Delta + \Delta\Delta = 0$.

- sum of green parts is 0:
$$x\,Sq^a Sq^b(y) + x \sum_{c} s(c)\, Sq^{a+b-c} Sq^c(y) = x\,R_{a,b}(y) \overset{\text{Assumption.}}{=} 0$$

- sum of red parts is 0:
$$x^4\left(Sq^{a-2} Sq^{b-1}(y) + \sum_{c} s(c)\, Sq^{a+b-c-2} Sq^{c-1}(y)\right)$$
$$\underset{a-2<2(b-1)}{\overset{\text{Assumption} +}{=}} x^4 \underbrace{\left(\sum_{c'} \binom{b-c'-2}{a-2-2c'} Sq^{a+b-c'-3} Sq^{c'}(y) + \sum_{c} s(c+1)\, Sq^{a+b-c-3} Sq^c(y)\right)}_{=0} = 0$$

Remains to show: obtained from $R_{a-1,b}$ (replaces $Sq^{a-1}Sq^b(y)$)

$$Sq^a Sq^{b-1}(y) + \overbrace{\sum_c{}' \binom{b-c-1}{a-2c-1} Sq^{a+b-c-1} Sq^c(y)}$$

$$= \sum_c{}' s(c)\, Sq^{a+b-c-1} Sq^c(y) + \sum_c{}' s(c)\, Sq^{a+b-c} Sq^{c-1}(y)$$

Case 1: $a = 2b-2$.

- $s(c) = \binom{b-c-1}{2(b-c-1)} = \begin{cases} 1 & b-c-1=0 \ (\Leftrightarrow c=b-1) \\ 0 & \text{else} \end{cases}$

RHS: $Sq^a Sq^{b-1}(y) + Sq^{a+1} Sq^{b-2}(y)$

- $\binom{b-c-1}{a-2c-1} = \binom{b-c-1}{2(b-c-1)-1} = \begin{cases} 1 & b-c-1=1 \ (\Leftrightarrow c=b-2) \\ 0 & \text{else} \end{cases}$

LHS: $Sq^a Sq^{b-1}(y) + Sq^{a+1} Sq^{b-2}(y)$

Case 2: $a = 2b-1$

Similar, Exercise

Case 3. $a < 2b-2$ comes from $R_{a,b-1}$ (since $a < 2(b-1)$)

$$\overbrace{\sum_c{}' \binom{b-c-2}{a-2c} Sq^{a+b-c-1} Sq^c(y)} + \sum{}' \binom{b-c-1}{a-2c-1} Sq^{a+b-c-1} Sq^c(y)$$

$$= \sum_c{}' \binom{b-c-1}{a-2c} Sq^{a+b-c-1} Sq^c(y) + \underbrace{\sum_c{}' \binom{b-c-1}{a-2c} Sq^{a+b-c} Sq^{c-1}(y)}_{= \sum_c{}' \binom{b-c-2}{a-2c-2} Sq^{a+b-c-1} Sq^c(y)}$$

index shift

Remains to check: $\binom{b-c-2}{a-2c} + \binom{\overbrace{b-c-1}^{=:p}}{\underbrace{a-2c-1}_{=:q}} + \binom{b-c-1}{a-2c} + \binom{b-c-2}{a-2c-2} = 0 \pmod 2$

$$\Leftrightarrow \binom{p-1}{q+1} + \binom{p}{q} + \binom{p}{q+1} + \binom{p-1}{q-1} = 0$$

This follows from

$$\binom{p}{q} = \binom{p-1}{q-1} + \binom{p-1}{q}$$

(Pascal triangle) ∎