

A cohomology operation of type (G, n, H, m) is a family of maps (of sets)

$$\Theta_X: H^n(X, G) \rightarrow H^m(X, H)$$

for all top. spaces X satisfying naturality:

$$\begin{array}{ccc} H^n(X, G) & \xrightarrow{\Theta_X} & H^m(X, H) \\ \uparrow f^* & \uparrow & \uparrow f^* \\ H^n(Y, G) & \xrightarrow{\Theta_Y} & H^m(Y, H) \end{array} \quad \forall f: X \rightarrow Y$$

$\mathcal{O}(G, n, H, m)$ set of cohomology op. of type (G, n, H, m)

Ex. "Cup product"

Type: $(\mathbb{R}, n, \mathbb{R}, 2n)$

comm. ring
w/ 1

$$\Theta_X: H^n(X, \mathbb{R}) \rightarrow H^{2n}(X, \mathbb{R})$$

$$X \longmapsto X^2$$

In this seminar: $\mathcal{O}(\mathbb{Z}/2, n, \mathbb{Z}/2, n+1)$

Thm 1 There exist natural $\mathbb{Z}/2$ -lin. maps

$$Sq^i: H^n(X, \mathbb{Z}/2) \rightarrow H^{n+i}(X, \mathbb{Z}/2) \quad \forall i \geq 0, n \geq 0$$

"Steenrod squares"

s.t.

(i) $Sq^0 = \text{id}$

(ii) $Sq^i(x) = 0 \quad \forall x \in H^n(X, \mathbb{Z}/2), i > n$

(iii) $Sq^n(x) = x^2 \quad \forall x \in H^n(X, \mathbb{Z}/2)$

(iv) $Sq^n(xy) = \sum_{i+j=n} Sq^i(x) Sq^j(y)$ "Cartan formula"

(v) $Sq^a \circ Sq^b = \sum_c \binom{b-c-1}{a-2c} Sq^{a+b-c} \circ Sq^c$, $a \leq 2b$
comp. of maps taken mod 2 "Adem relations"

(vi) ...

Proof Existence (Talk 2+3), clean statement and proof of properties (Talk 4+5)

$$A := \mathbb{Z}/2 \langle Sq^i \mid i \geq 0 \rangle / \text{Adem relations} \quad \left(\begin{array}{l} \text{graded algebra} \\ \text{deg}(Sq^i) = i \end{array} \right) \quad \text{"(mod-2) Steenrod algebra"}$$

$$\leadsto A \subset H^*(X, \mathbb{Z}/2) \quad \text{cohomology ring becomes a } A\text{-module} \quad \text{(Talk 7+8)}$$

Motivation: There exist spaces whose cohomologies are isomorphic as rings, but not as A -modules (Exercise?) \rightarrow Distinguish more spaces!

Ex 1 $H^*(\mathbb{R}P^\infty, \mathbb{Z}/2) \cong \mathbb{Z}/2[\alpha]$ $\deg(\alpha) = 1$

Describe $H^*(\mathbb{R}P^\infty, \mathbb{Z}/2)$ as A -module!

$Sq := Sq^0 + Sq^1 + Sq^2 + \dots$ (property (ii))

$H^*(X, \mathbb{Z}/2) \xrightarrow{Sq} H^*(X, \mathbb{Z}/2)$

$Sq_f(x) Sq_f(y) \stackrel{\text{Def}}{=} \left(\sum_{i \geq 0} Sq^i(x) \right) \left(\sum_{j \geq 0} Sq^j(y) \right)$

$= \sum_{n \geq 0} \sum_{i+j=n} Sq^i(x) Sq^j(y)$

$\stackrel{\text{Catalan}}{=} \sum_{n \geq 0} Sq^n(xy) \stackrel{\text{Def}}{=} Sq_f(xy)$

$Sq(\alpha^n) = (Sq(\alpha))^n = (\alpha + \alpha^2)^n = \sum_{i=0}^n \binom{n}{i} \alpha^{n+i}$

$Sq^0(\alpha) = \alpha$

$Sq^1(\alpha) = \alpha^2$

$Sq^{>1}(\alpha) = 0$ (prop. (ii))

$\Rightarrow Sq^i(\alpha^n) = \binom{n}{i} \alpha^{n+i}$

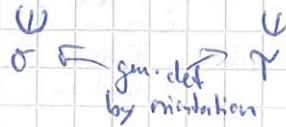
Application (Talk 6):

S^n oriented n -sphere, $n \geq 2$

$f: S^{2n-1} \rightarrow S^n$

$K = S^n \cup_f e^{2n}$ oriented $2n$ -cell

$H^0(K, \mathbb{Z}) \cong H^n(K, \mathbb{Z}) \cong H^{2n}(K, \mathbb{Z}) \cong \mathbb{Z}$



$\Rightarrow \sigma^2 = \underbrace{H(f)}_{\in \mathbb{Z}}$

"Hopf invariant"

Thm 2 $H(f) = 1 \Rightarrow n = 2^k$ for some k .

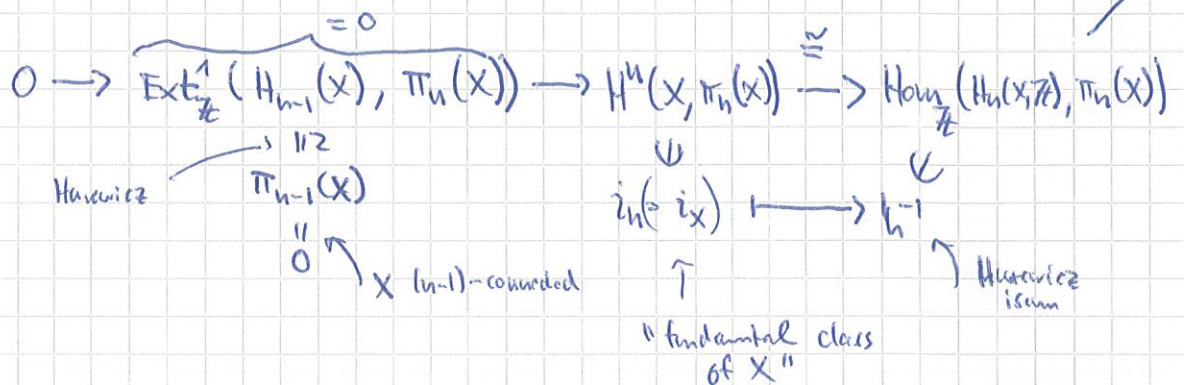
Q: Can we classify coboundary operations $\mathcal{O}(G, n, H, m)$?

"Hurewicz homom." $h: \pi_i(X) \rightarrow H_i(X, \mathbb{Z})$ fix $u \in H_i(S^i, \mathbb{Z}) \cong \mathbb{Z}$ a generator
 $[f: S^i \rightarrow X] \mapsto f_*(u)$, $i \geq 1$ (2)

X $(n-1)$ -connected (all homotopy groups $\pi_i(X) \cong 0$, $i \leq n-1$)
 $\Rightarrow h$ is isom. $\forall i \in \mathbb{N}$ (surj. for $i+1$)

"Univ. Coeff. thm." $0 \rightarrow \text{Ext}_{\mathbb{Z}}^1(H_{n-1}(X, \mathbb{Z}), G) \rightarrow H^n(X, G) \rightarrow \text{Hom}_{\mathbb{Z}}(H_n(X, \mathbb{Z}), G) \rightarrow 0$
 exact seq. for any X .

X $(n-1)$ -connected.



$K(G, n)$ top. space s.t. (i) $\pi_n(K(G, n)) \cong G$, zero else
 "Eilenberg-MacLane-space" (ii) has homotopy type of CW-complex

Fact. $K(G, n)$ exists and is unique up to homotopy eq. [MT68, p.5]

Ex: $K(\mathbb{Z}, 1) \cong S^1$, $K(\mathbb{Z}/2, 1) \cong \mathbb{R}P^\infty$

$\text{Hom}_{\mathbb{Z}}[X, K(G, n)] \xrightarrow{\cong} H^n(X, G)$ bij. "Brown representability"
 $[f] \mapsto f^*(z_n)$
 homotopy classes of maps \rightarrow Eilenberg-MacLane spectrum represents sing. cohomology!

Thm 4 $\mathcal{O}(G, n, H, m) \xrightarrow{\cong} H^m(K(G, n), H)$
 $\oplus \mapsto \oplus(z_n)$

Proof: [MT68, pp. 6-10]

Proof: $\varphi \in H^m(K(G, n), H) \rightsquigarrow H^m(X, G) \rightarrow H^m(X, H)$
 $u \mapsto f^*(\varphi)$
 unique $f: X \rightarrow K(G, n)$ s.t. $[f]$ corresponds to u under isom. in Thm 3.

Exercise: We have inverse maps!

In our case:

$$\mathcal{G}(\mathbb{Z}/2, n, \mathbb{Z}/2, \text{ufi}) \xrightarrow{\cong} H^{\text{ufi}}(K(G, n), \mathbb{Z}/2)$$

Compute $H^*(K(G, n), \mathbb{Z}/2)$ to understand cohomology operations!
(Talks 12+13)

Computation uses machinery of spectral sequences!
(Talks 9, 10, 11).