A cohomology operation of type (G, n, H, m) is a family of maps (of sets)

ab. groups — nonnegative integers

$$\Theta_X : H^n(X, G) \longrightarrow H^m(X, H)$$

for all top. spaces X satisfying naturality:

$$\begin{array}{ccc} H^n(X,G) & \xrightarrow{\Theta_X} & H^m(X,H) \\ f^* \uparrow & \circlearrowleft & \uparrow f^* \\ H^n(Y,G) & \xrightarrow[\Theta_Y]{} & H^m(Y,H) \end{array} \qquad \forall f: X \longrightarrow Y$$

$\mathcal{O}(G,n,H,m)$ set of cohomology op. of type (G,n,H,m)

Ex. "Cup product"

Type: $(R, n, R, 2n)$ comm. ring w/ 1

$$\Theta_X : H^n(X,R) \longrightarrow H^{2n}(X,R)$$
$$x \longmapsto x^2$$

In this seminar: $\mathcal{O}(\mathbb{Z}/2, n, \mathbb{Z}/2, n+i)$

Thm 1 There exist natural $\mathbb{Z}/2$-lin. maps

$$Sq^i : H^n(X, \mathbb{Z}/2) \longrightarrow H^{n+i}(X, \mathbb{Z}/2) \quad \forall\, i \geq 0,\ n \geq 0$$ "Steenrod squares"

s.th.

(i) $Sq^0 = id$

(ii) $Sq^i(x) = 0 \quad \forall x \in H^n(X, \mathbb{Z}/2)$, $i > n$

(iii) $Sq^n(x) = x^2 \quad \forall x \in H^n(X, \mathbb{Z}/2)$

(iv) $Sq^n(xy) = \sum'_{i+j=n} Sq^i(x)\, Sq^j(y)$ "Cartan formula"

(v) $\underbrace{Sq^a Sq^b}_{\text{compos. of maps}} = \sum'_c \underbrace{\binom{b-c-1}{a-2c}}_{\text{taken mod 2}} Sq^{a+b-c}\, Sq^c$, $a < 2b$ "Adem relations"

(vi) ...

Proof Existence (Talk 2+3), Clean statement and proof of properties (Talk 4+5)

$$\mathcal{A} := \mathbb{Z}/2\langle Sq^i \mid i \geq 0 \rangle / \text{Adem relations}$$ (graded algebra, $\deg(Sq^i) := i$) "(mod-2) Steenrod algebra"

$\rightsquigarrow \mathcal{A} \circlearrowright H^*(X, \mathbb{Z}/2)$ cohomology ring becomes a $\mathcal{A}$ module (Talk 7+8)

Motivation: There exist spaces whose cohomologies ~~rings~~ are isomorphic as rings, but not as $\mathcal{A}$-modules (Exercise?) $\longrightarrow$ Distinguish more spaces!

Ex. $H^*(\mathbb{RP}^\infty, \mathbb{Z}/2) \cong \mathbb{Z}/2[\alpha] \quad \deg(\alpha) = 1$

Describe $H^*(\mathbb{RP}^\infty, \mathbb{Z}/2)$ as $\mathcal{A}$-module!

$Sq := Sq^0 + Sq^1 + Sq^2 + \dots$ (property (iii))

$$H^*(X, \mathbb{Z}/2) \xrightarrow{Sq} H^*(X, \mathbb{Z}/2)$$

$$Sq(x)\,Sq(y) \overset{\text{Def}}{=} \Big(\sum_{i\geq 0}{}^{'} Sq^i(x)\Big)\Big(\sum_{j\geq 0}{}^{'} Sq^j(y)\Big)$$

$$= \sum_{n\geq 0}{}^{'} \sum_{i+j=n}{}^{'} Sq^i(x)\,Sq^j(y)$$

$$\overset{\text{Cartan}}{=} \sum_{n\geq 0}{}^{'} Sq^n(xy) \overset{\text{Def}}{=} Sq(xy)$$

$$Sq(\alpha^n) = (Sq(\alpha))^n = (\alpha + \alpha^2)^n = \sum_{i=0}^{n}{}^{'} \binom{n}{i} \alpha^{n+i}$$

$Sq^0(\alpha) = \alpha$

$Sq^1(\alpha) = \alpha^2$

$Sq^{>1}(\alpha) = 0$ (prop. (ii))

$$\Rightarrow \quad Sq^i(\alpha^n) = \binom{n}{i} \alpha^{n+i}$$

Application (Talk 6): S^n oriented n-sphere, $n \geq 2$

$f: S^{2n-1} \longrightarrow S^n$

$K = S^n \cup_f e^{2n}$ ← oriented $2n$-cell

$$H^0(K,\mathbb{Z}) \cong H^n(K,\mathbb{Z}) \cong H^{2n}(K,\mathbb{Z}) \cong \mathbb{Z}$$

$\sigma \in H^n(K,\mathbb{Z})$, $\tau \in H^{2n}(K,\mathbb{Z})$ gen. def by orientation

$$\Rightarrow \quad \sigma^2 = \underbrace{H(f)}_{\in \mathbb{Z}}\,\tau$$

"Hopf invariant"

Thm. 2 $H(f) = 1 \Rightarrow n = 2^k$ for some k.

Q: Can we classify cohomology operations $\mathcal{O}(G,n,H,m)$?

"Hurewicz homom." $h: \pi_i(X) \longrightarrow H_i(X,\mathbb{Z})$ fix $u \in H_i(S^i,\mathbb{Z}) \cong \mathbb{Z}$ a generator

$[f: S^i \to X] \mapsto f_*(u)$, $i \geq 1$ (2)

X $(n-1)$-connected (all homotopy groups $\pi_i(X) \cong 0$, $i \leq n-1$)
$\Rightarrow$ h is isom. $\forall i \leq n$ (surj. for $i+1$)

"Univ. coeff. thm." $0 \longrightarrow \mathrm{Ext}^1_{\mathbb{Z}}(H_{n-1}(X,\mathbb{Z}), G) \longrightarrow H^n(X,G) \longrightarrow \mathrm{Hom}_{\mathbb{Z}}(H_n(X,\mathbb{Z}), G) \longrightarrow 0$

exact seq. for any X.

X $(n-1)$-connected.

$$0 \longrightarrow \overbrace{\mathrm{Ext}^1_{\mathbb{Z}}(H_{n-1}(X), \pi_n(X))}^{=0} \longrightarrow H^n(X,\pi_n(X)) \xrightarrow{\cong} \mathrm{Hom}_{\mathbb{Z}}(H_n(X,\mathbb{Z}), \pi_n(X)) \nearrow 0$$

Hurewicz: $H_{n-1}(X) \cong \pi_{n-1}(X) = 0$ (X $(n-1)$-connected)

$i_n (= i_X) \longmapsto h^{-1}$

"fundamental class of X" — Hurewicz isom

$K(G,n)$ top. space s.th. (i) $\pi_n(K(G,n)) \cong G$, zero else
"Eilenberg-Maclane-Space" (ii) has homotopy type of CW-complex

Fact. $K(G,n)$ exists and is unique up to homotopy eq. [MT68, p.5]

Ex: $K(\mathbb{Z},1) \simeq S^1$, $K(\mathbb{Z}/2,1) \simeq \mathbb{RP}^\infty$

Thm 3 $[X, K(G,n)] \xrightarrow{\cong} H^n(X,G)$ bij. "Brown representability"

$[f] \longmapsto f^*(i_n)$

homotopy classes of maps

Eilenberg-Maclane spectrum represents sing. cohomology!

Thm 4 $\mathcal{O}(G,n,H,m) \xrightarrow{\cong} H^m(K(G,n),H)$

$\Theta \longmapsto \Theta(i_n)$

Proof: [MT68, pp. 6-10]

Proof. $\varphi \in H^m(K(G,n),H) \rightsquigarrow H^n(X,G) \longrightarrow H^m(X,H)$

$u \longmapsto f^*(\varphi)$

unique $f: X \to K(G,n)$ s.th. $[f]$ corresponds to u under isom. in Thm 3.

Exercise: We have inverse maps!

In our case:

$$\mathcal{O}(\mathbb{Z}/2, n, \mathbb{Z}/2, n+i) \xrightarrow{\cong} H^{n+i}(K(G,n), \mathbb{Z}/2)$$

Compute $H^*(K(G,n), \mathbb{Z}/2)$ to understand cohomology operations!
(Talks 12+13)

Computation uses machinery of spectral sequences!
(Talks 9,10,11).