Talk 9. The Serre Spectral Sequence.

Xing Gu.

9.1. A Toy Example.

Throughout this section, let $\hat{C}, \hat{H} \cdots$ denote simplicial stuff, and $C, H \cdots$ singular stuff.

9.1.1 Exercise: Without using simplicial homology, compute the homology of spheres:

$$H_k(S^n) \cong \begin{cases} \mathbb{Z}, & k=n, \\ 0, & \text{otherwise}. \end{cases}$$

Hint: Trivial for $n=0$. Inductively, use Mayer-Vietoris sequence to complete the argument.

9.1.2. Question: Let K be a simplicial complex. $|K|$ denotes its geometric realization. Why do we have

$$\hat{H}_*(K) \cong H_*(|K|)\ ?$$

Let $K^{(n)}$ be the n-th skeleton of K. Then we have the long exact sequence

$$\cdots \to H_k(|K^{(n-1)}|) \to H_k(|K^{(n)}|) \to H_k(|K^{(n)}|, |K^{(n-1)}|) \xrightarrow{\partial} H_{k-1}(|K^{(n-1)}|) \to \cdots$$

9.1.3 Lemma: $H_*(|K^{(n)}|, |K^{(n-1)}|) \cong \Sigma_\alpha H_*(S^n)$, where α runs over all n-simplices of K. Moreover, each $H_n(S^n)$ is generated by the identification $\Delta^n \xrightarrow{\cong} \Delta^n_\alpha$, where Δ^n_α is an n-simplex of K.

Proof: Exercise. $\blacksquare$

Consider the composition.

$$\begin{array}{ccccc} d_n: H_n(|K^{(n)}|, |K^{(n-1)}|) & \xrightarrow{\partial} & H_{n-1}(|K^{(n-1)}|) & \to & H_{n-1}(|K^{(n-1)}|, |K^{(n-2)}|) \\ \wr\| & & & & \wr\| \\ \sum_\alpha H_n(S^n) & & & & \sum_\beta H_{n-1}(S^{n-1}) \\ \wr\| & & & & \wr\| \\ \hat{C}_n(K) & & \xrightarrow{\partial_n} & & \hat{C}_{n-1}(K) \end{array}$$

9.1.4. Proposition The diagram above commutes.

Proof: First we verify ∂:

$$\begin{array}{ccccccc} C_n(|K^{(n-1)}|) & \to & C_n(|K^{(n)}|) & \to & C_n(|K^{(n)}|, |K^{(n-1)}|) & \to & 0 \\ \downarrow \partial_n & & \downarrow \partial_n & & \downarrow \partial_n & & \\ 0 \to C_{n-1}(|K^{n-1}|) & \to & C_{n-1}(|K^{n}|) & \to & C_{n-1}(|K^{(n)}|, |K^{(n-1)}|) & & \end{array}$$

~~Check: Everything in $C_n(|K^{(n)}|, |K^{(n-1)}|)$ is a cycle.~~

Let $\Delta^n \xrightarrow{\chi_\alpha} \Delta^n_\alpha$ be the singular chain identifying Δ^n to Δ^n_α, an n-simplex of K. Let $\overline{\chi_\alpha} \in C_n(|K^n|, |K^{n-1}|)$ be the projection of χ_α.

Check: $\overline{\chi_\alpha}$ is a cycle. Moreover, $H_n(|K^{(n)}|, |K^{(n-1)}|)$ is generated by $\overline{\chi_\alpha}$'s.

On the other hand,

$$\partial_n(\chi_\alpha) = \sum_{i=0}^n (-1)^i \chi_\alpha|_{F_i},$$

where F_i is the i-th face of Δ^n. Then $\chi_{\alpha,i} := \chi_\alpha|_{F_i}$

are $(n-1)$-simplices of K. i.e.

$$\partial_n(x_\alpha) \in C_{n-1}(|K^{(n-1)}|).$$

$\Longrightarrow$

$$\cancel{\partial_n}\ \partial([\bar{x}_\alpha]) = \left[\sum_{i=0}^{n} (-1)^i x_{\alpha,i}\right] \in H_{n-1}(|K^{(n-1)}|).$$

Remark: The $x_{\alpha,i}$'s are not cycles, ~~whence~~ hence there is no $[x_{\alpha,i}]$!

Next, we consider.

$$H_{n-1}(|K^{(n-1)}|) \longrightarrow H_{n-1}(|K^{(n-1)}|, |K^{(n-2)}|).$$

Notice that $\bar{x}_{\alpha,i} \in C_{n-1}(|K^{(n-1)}|, |K^{(n-2)}|)$ are cycles, it suffices to remark that the homomorphism above does the following:

$$\partial([\bar{x}_\alpha]) = \left[\sum_{i=0}^{n} (-1)^i x_{\alpha,i}\right] \longmapsto \sum_{i=0}^{n} (-1)^i [\bar{x}_{\alpha,i}] \in H_{n-1}(|K^{(n-1)}|, |K^{(n-2)}|)$$

This completes the proof. ■

Remark: It is automatically verified that $d \circ d = 0$.

In other words, we've identified the chain complexes $\{H_n(|K^{(n)}|, |K^{(n-1)}|), d_n\}$ and $\{\hat{C}_n(K), \partial_n\}$. It remains to show that the homology of the former is $H_*(|K|)$.

In the ~~fo~~ following diagram, all the slant sequences are exact.

$$
\begin{tikzcd}
 & & & & H_n(|K^{(n+1)}|, |K^{(n)}|) = 0. \\
H_n(|K^{(n-1)}|) = 0. \arrow[dr] & & & H_n(|K^{(n+1)}|) \arrow[ur] & \\
 & & H_n(|K^{(n)}|) \arrow[ur, "i_n"] \arrow[dr, "j_n"] & & \\
 & H_{n+1}(|K^{(n+1)}|, |K^{(n)}|) \arrow[ur, "\partial_{n+1}"] \arrow[rr, "d_{n+1}"] & & H_n(|K^{(n)}|, |K^{(n-1)}|) \arrow[rr, "d_n"] \arrow[dr, "\partial_n"] & & H_{n-1}(|K^{(n-1)}|, |K^{(n-2)}|) \\
 & & & & H_{n-1}(|K^{(n-1)}|) \arrow[ur, "j_{n-1}."] & \\
 & & & H_{n-1}(|K^{(n-2)}|) = 0. \text{ (Why?)} \arrow[ur] & &
\end{tikzcd}
$$

$$\operatorname{Ker} d_n = \operatorname{Ker}(j_{n-1} \circ \partial_n) \overset{j_{n-1} \text{ is injective.}}{\cong} \operatorname{Ker} \partial_n. = \operatorname{Im} j_n \overset{j_n \text{ is injective.}}{\cong} H_n(|K^{(n)}|)$$

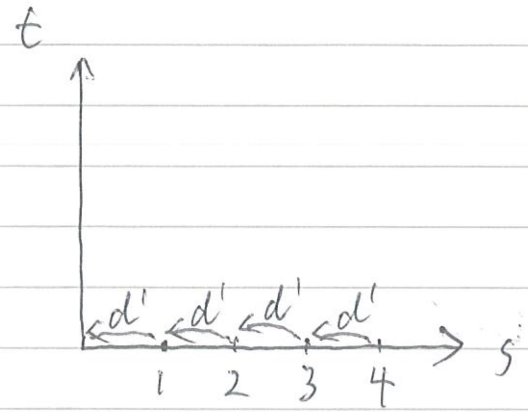
$$\operatorname{Im} d_{n+1} = \operatorname{Im}(j_n \circ \partial_{n+1}) \underset{j_n \text{ is injective}}{\cong} \operatorname{Im}(\partial_{n+1})$$

$$\Rightarrow \operatorname{Ker} d_n / \operatorname{Im} d_{n+1} \cong H_n(|K^{(n)}|) / \operatorname{Im} \partial_{n+1} \overset{i_n \text{ is surjective.}}{\cong} H_n(|K^{(n+1)}|) \overset{\text{Why?}}{\cong} H_n(|K|)$$

9.1.5 Figure.

Let $E^1_{s,t} := H_{s+t}(|K^{(s)}|, |K^{(s-1)}|)$, and $d^1_{s,t} = d_s$, then

" $\{E^r_{s,t}, d^r_{s,t}\}$ is a spectral sequence converging to $H_*(|K|)$ and collapsing at E^2-page "



§9.2 Spectral Sequences.

9.2.1 Definition: Let X (resp. C_*) be a topological space (resp. chain complex). A filtration on X (resp. C_*) is a series of subspaces (sub-complexes)

$X^{(0)} \subset X^{(1)} \subset \cdots$

(resp. $C_*^{(0)} \subset C_*^{(1)} \cdots\cdots$).

Remark: (i) The above is sometimes called an increasing filtration. One can re-index it to get a decreasing filtration.

(ii) A filtered space gives rise to a filtered chain complex by taking its singular chain.

9.2.2. Construction: Let $E^0_{s,t} := C^{(s)}_{s+t} / C^{(s-1)}_{s+t}$, then we have a induced boundary homomorphism.

$$d^0_{s,t} \; \cancel{d^{s,t}} : E^0_{s,t} \longrightarrow E^0_{s,t-1}$$

Let $E^1_{s,t} := H_{s+t}(E^0_{*,*}; d^0_{*,*})_s \cong \ker d^0_{s,t} / \operatorname{Im} d^0_{s,t+1}$

As in 9.1.5 Figure, there are another set of induced boundary homomorphisms.

(9.2.3) $d^1_{s,t} : E^1_{s,t} \longrightarrow E^1_{s-1,t}$.

However, the absence of inj./surj. conditions stops us from getting $H_*(C_*)$. Instead, we get another set of sub-quotients of $E^1_{*,*}$, denoted by $E^2_{*,*}$, with $d^2_{*,*}$ the induced boundary.

Inductively, we obtain $(E^r_{*,*}, d^r_{*,*})$ with

$E^r_{s,t} = \operatorname{Ker} d^{r-1}_{s,t} / \operatorname{Im} d^{r-1}_{s-r+1,\, t+r}$ and

(9.2.4)
$$d^r_{s,t} : E^r_{s,t} \longrightarrow E^r_{s-r,\, t+r-1}.$$

Remark: There are other ways to arrange the indices, resulting in different ~~...~~ forms of E and d.

9.2.5 Definition. (9.2.3), (9.2.4) and the text between them is a definition of spectral sequences. If for a pair (s,t), $E^r_{s,t} = E^{r+1}_{s,t}$ for $r \gg 0$, we say that the spectral sequence collapses at (s,t), and write $E^\infty_{s,t} = E^r_{s,t}$ for $r \gg 0$.

9.2.6 Explanation: We define the following sub-modules of C_*:

$$Z^r_{s,t} := \{ x \in C^{(s)}_{s+t} \mid d(x) \in C^{(s-r)}_{s+t-1} \}$$

$$B^r_{s,t} := \{ x \in C^{(s)}_{s+t} \mid x = d(y) \text{ for some } y \in C^{(s+r)}_{s+t+1} \}.$$

Then we have $Z^{r+1}_{s-1,t+1} \subset Z^r_{s,t}$. $B^{r+1}_{s-1,t+1} \subset B^r_{s,t}$. Let.

$$\overline{Z}^r_{s,t} := Z^r_{s,t} / Z^{r+1}_{s-1,t+1}, \quad \overline{B}^r_{s,t} := B^r_{s,t} / B^{r+1}_{s-1,t+1}$$

be sub-modules of $C^{(s)}_{s+t} / C^{(s-1)}_{s+t}$. Then we have

$$\overline{B}^0_{s,t} \subset \cdots \subset \overline{B}^r_{s,t} \subset \overline{B}^{r+1}_{s,t} \subset \cdots \subset \overline{B}^\infty_{s,t} \subset \overline{Z}^\infty_{s,t} \subset \cdots \overline{Z}^r_{s,t} \subset \cdots \overline{Z}^0_{s,t}.$$

We have $E^r_{s,t} \cong \overline{Z}^r_{s,t} / \overline{B}^r_{s,t}$. $E^\infty_{s,t} \cong H_{s+t}(C_*)^{(s)} / H_{s+t}(C_*)^{(s-1)}$.

$$\left(\begin{tikzcd} C^{(s-r)} \arrow[r, phantom, "\subset \cdots \subset"] & C^{(s)} \arrow[l, bend right, "d"'] \arrow[r, phantom, "\subset \cdots \subset"] & C^{(s+r)} \arrow[l, bend left, "d"] \end{tikzcd} \right)$$

9.2.7. A theorem about spectral sequences often takes the form:

"There is a spectral sequence $(E^*_{**}, d^{(*)}_{**})$ converging to (something desirable), such that

$E^2_{s,t} \cong$ (Something computable, homotopy invariant in a proper sense.)"

§9.3. Serre Spectral Sequence.

9.3.1. ~~Definition:~~ Let $p: X \to B$ be a map of spaces, such that for any CW-complex K, the following lifting problem has a solution up to homotopy:

$$\begin{tikzcd} K \arrow[r] \arrow[d, "i_0"'] & X \arrow[d, "p"] \\ K\times I \arrow[r] \arrow[ur, dashed] & B \end{tikzcd}$$

Here, $I = [0,1]$, $i_0(k) = (k,0)$, and the horizontal maps are arbitrary as long as the square commutes. Such a map p is called a <u>Serre fibration</u>.

9.3.2 Definition / Theorem: $F_b := p^{-1}(b)$ is called a fiber of p over b.

The lifting property above induces a functor from the fundamental groupoid $\pi(B)$ of B to the full subcategory of $Ho(\mathcal{T}op)$ with objects $\{F_b\}_{b\in B}$.

In particular, all fibers are homotopy equivalent to each other, if B is path-connected.

Remark: Usually we pick a point $b_0 \in B$, and write $F := F_{b_0}$ and say that $F \to X \to B$ is a fiber sequence. More generally, we say that $F' \to X' \to B'$ is a fiber sequence up to homotopy if we have a homotopy commutative diagram

$$\begin{tikzcd} F \arrow[r] \arrow[d] & X \arrow[r] \arrow[d] & B \arrow[d] \\ F' \arrow[r] & X' \arrow[r] & B' \end{tikzcd}$$

where $F \to X \to B$ is a fiber sequence. We will not distinguish fiber sequences and homotopy fiber sequences from now on

9.3.3 Theorem: Let B be a path-connected topological space, $F \to X \to B$ a fiber sequence. Let R be a commutative unit ring.

The homotopy groupoid $\pi(B)$ give rise to a so-called "local coefficient system", taking value ~~at~~ $H_*(F_b; R)$ at each $b \in B$. We denote it by $\underline{H_*}(F; R)$.

Then there is a spectral sequence (E, d) converging to $H_*(X; R)$ s.t.

$$E^2_{s,t} \cong H_{s+t}(B; \underline{H_t(F; R)}).$$

When B is ~~path~~ simply-connected.

$$E^2_{s,t} \cong H_{s+t}(B; H_t(F; R)).$$

Dually, there is an obvious cohomological version.