Talk 10. Transgression and the cohomology spectral sequence of a fibration.

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§1. Review.

Last talk: $p: X \to B$ a map of spaces, such that for any CW-complex K, the following homotopy lifting problem has a solution:

$$\begin{array}{ccc} K & \longrightarrow & X \\ {\scriptstyle i_0}\downarrow & & \downarrow{\scriptstyle p} \\ K\times I & \longrightarrow & B \end{array}$$

Then p is called a Serre fibration. Also recall a fiber sequence $F \to X \to B$.

Example: A fiber bundle $p: X \to B$ is a Serre fibration. If the fiber is F, then $F \to X \to B$ is a fiber sequence.

Theorem (9.3.2, last talk). B - simply connected, $F \to X \to B$ a fiber sequence. Then there is a spectral sequence $(E^r_{s,t}, d^r_{s,t})$ such that $E^2_{s,t} \cong H_s(B, H_t(F))$, $d^r_{s,t}: E^r_{s,t} \to E^r_{s-r,t+r-1}$, and converges to $H_{s+t}(X)$.

Dually, we have a spectral sequence $(E_r^{s,t}, d_r^{s,t})$, such that $E_2^{s,t} \cong H^s(B; H^t(F))$. $d_r^{s,t}: E_r^{s,t} \to E_r^{s+r,t-r+1}$, and converges to $H^{s+t}(X)$.

Remark: There is a more general version of the theorem for B not necessarily simply connected. Google "local coefficient systems".

10.1.1. Corollary: If the coefficient ring is a field, then $E^2_{s,t} \cong H_s(B) \otimes H_t(F)$, $E_2^{s,t} \cong H^s(B) \otimes H^t(F)$.

§2. Cohomological Serre Spectral Sequence.

Thoughout this section, $F \to X \xrightarrow{p} B$ is a fiber sequence; $(E_r^{s,t}, d_r^{s,t})$ is the associated cohomological Serre spectral sequence with coefficients in a ~~ring~~ field. K. B is simply-connected.

Theorem 10.2.1. (1) The bigraded K-module $E_2^{*,*}$ has a K-algebra structure inherited from $H^*(B)$ and $H^*(F)$.

(2). Each $E_r^{*,*}$, $r>2$, inductively has a K-algebra structure inherited from $E_{r-1}^{*,*}$.

(3) For $r \geq 2$, $d_r^{s,t}$ satisfies the Leibniz rule:

$$d_r(a \cdot b) = d_r(a) \cdot b + (-1)^{\deg(a)} a \cdot d_r(b).$$

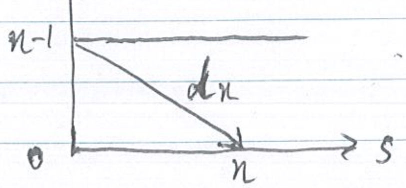
(Is this really non-homogeneous?).

(4). $E_\infty^{s,0} \cong \mathrm{Im}\{p^* : H^s(B) \to H^s(X)\}$ (True for any coeff.)

$E_\infty^{0,t} \cong \mathrm{Im}\{H^t(X) \longrightarrow H^t(F)\}$

(Notice that $E_\infty^{s,0}$ is a quotient of $E_2^{s,0}$, wherea $E_\infty^{0,t}$ is a sub K-module of $E_2^{0,t}$)

10.2.2. Example (Gysin's sequence). Let $X \xrightarrow{p} B$ be a $(n-1)$-sphere bundle, then $F = S^{n-1}$. The Serre spectral sequence looks like



$$E_2^{s,t} \cong \begin{cases} H^s(B) & s=0 \text{ or } n-1, \\ 0 & \text{otherwise.} \end{cases}$$

In particular, $E_2^{0,n-1} \cong E_3^{0,n-1} \cong \cdots \cong E_n^{0,n-1} \cong H^0(B) \cong K$. generated by $1 \otimes u$.

Let $e = d_n(1 \otimes u) \in H^n(X)$. Then by (3) of (10.2.1),

$$d_n^{s,n-1} : \underset{\substack{\text{\rotatebox{90}{$\cong$}} \\ H^s(B)}}{E_2^{s,n-1}} \longrightarrow \underset{\substack{\text{\rotatebox{90}{$\cong$}} \\ H^{s+n}(B)}}{E_2^{s+n,0}}$$

is the multiplication by u.

Furthermore, the figure above shows that the spectral sequence collapses at the $(n+1)$-th page. In particular, we have

$$E_\infty^{s,n-1} \cong E_{n+1}^{s,n-1} = \operatorname{Ker}\{d_n^{s,n-1}: E_n^{s,n-1} \to E_n^{s+n,0}\},$$

$$E_\infty^{s,0} \cong E_\infty^{s,0} = \operatorname{Coker}\{d_n^{s-n,n-1}: E_n^{s-n,n-1} \to E_n^{s,0}\}.$$

Therefore, we have the following exact sequence

$$0 \to E_\infty^{s,n-1} \to H^s(B) \xrightarrow{ue} H^{s+n}(B) \to E_\infty^{s+n,0} \to 0.$$

On the other hand, we have another exact sequence

$$0 \to E_\infty^{s,0} \to H^s(X) \to E_\infty^{s-n+1,n-1} \to 0.$$

The two exact sequences glued together give

$$\cdots \to H^{s+n-1}(X) \to H^s(B) \xrightarrow{ue} H^{s+n}(B) \to H^{s+n}(X) \to \cdots$$

When $p: X \to B$ is ~~the~~ sphere bundle associated ~~to~~ an n-vector bundle, e is called the Euler class of the ~~sphere~~ vector bundle.

Remark: The argument holds for B non-simply connected.

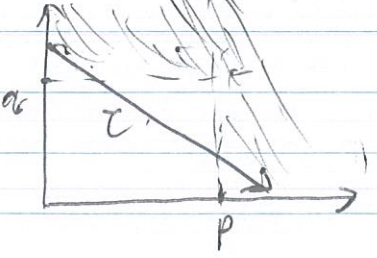
10.2.3. Exercise: Use 10.2.2 to verify the K-algebra structure of $H^*(\mathbb{C}P^n)$ (Hint: Consider the sphere bundle $S^1 \to S^{2n+1} \to \mathbb{C}P^n$.)

10.2.4. Proposition: If B (resp. F) is $(p-1)$- (resp. $(q-1)$)-connected, then we have an ~~long~~ exact sequence.

$$\cdots \to H^{p+q-2}(F) \xrightarrow{\tau} H^{p+q-1}(B) \xrightarrow{p^*} H^{p+q-1}(X) \xrightarrow{i^*} H^{p+q-1}(F)$$

Terminates!

Proof:



Consider $\tau = d_n^{0,n-1} : E_n^{0,n-1} \to E_n^{n,0}$ in general: Notice that $E_n^{0,n-1}$ is a sub-module of $E_2^{0,n-1} \cong H^{n-1}(F)$, and $E_n^{n,0}$ a quotient of $H^n(B)$

10.2.5. Definition: An element of $H^{n-1}(F)$ is called transgressive if it is in $E_n^{0,n-1}$. The homomorphism $\tau = d_n^{0,n-1}$ is called the transgression.

In the next theorem, we take $K = \mathbb{Z}/2$.

10.2.6 Theorem: Let $x \in H^{n-1}(F)$. If x is transgressive, then so is $Sq^i(x)$, $\forall i \geq 0$. Furthermore, the transgression commutes with the Steenrod operations. i.e. $\tau(Sq^i(x)) = Sq^i(\tau(x))$.

Remark: (1) This is true for $K = \mathbb{Z}/p$ and the Steenrod power operations, for all primes p.

(2) The transgression could be interpreted by ~~as the "inverse" of~~ the following homomorphism $(n > 0)$

$$\begin{tikzcd} H^n(B) \cong H^n(B, pt) \arrow[r, "p^*"] & H^n(X, F) \\ & H^{n-1}(F) \arrow[u, "\delta"] \arrow[ul, dashed, bend left, "\tau"] \end{tikzcd}$$

τ is then defined on $x \in H^{n-1}(F)$ such that $\delta(x)$ is in the image of p^*. This partly explains 10.2.6.

§3. The path Space Fibration.

10.3.1 Definition: Fix a base point $* \in B$. Let
$$PB := \{\sigma : I \to B \mid \sigma(1) = *\}$$
with a suitable topology.

10.3.2 Definition/Theorem: The map
$$p : PB \to B \quad \sigma \mapsto \sigma(0) \text{ is a Serre fibration.}$$
with fiber $p^{-1}(*) \cong \Omega B. = \{\sigma : I \to B \mid \sigma(0) = \sigma(1) = *\}$.

10.3.2 Proposition: PB is contractible.

10.3.3 Proposition: $\pi_i(B) \cong \pi_{i-1}(\Omega B)$ $i \geq 1$.

Therefore, the Serre spectral sequence of the fiber sequence $\Omega B \to PB \to B$. satisfies

$$E_\infty^{s,t} \cong \begin{cases} K, & s=t=0 \\ 0, & \text{otherwise}. \end{cases}$$

We use this information to study B or ΩB.

10.3.4. Proposition: $\Omega K(\pi, n) \simeq K(\pi, n-1)$

Proof: See 10.3.3.

10.3.5. Therefore, we have a fiber sequence

$$K(\pi, n-1) \to X \to K(\pi, n)$$

where X is contractible. This will be the topic of the next two talks.

10.3.6. Exercise: Compute $H^*(\Omega S^n)$.