

The mod-2 Steenrod algebra

Recall: k field

• V k -v.s.p., grading on V is a decomposition $V = \bigoplus_{i \geq 0} V_i$,
 V together with grading is called graded

vector subspaces

• $V = \bigoplus_{i \geq 0} V_i$, $W = \bigoplus_{i \geq 0} W_i$ graded k -v.s.p.

(i) k -lin map $f: V \rightarrow W$ is graded if $f(V_i) \subseteq W_i$, $\forall i \geq 0$.

(ii) $V \otimes W$ is equipped w/ grading

$(V \otimes W)_i =$ "vector subspace generated by $v \otimes w$
 s.th. $|v| + |w| = i$ "

Def 1 associative, graded k -alg w/ 1 consists of data:

• graded k -v.s.p $A = \bigoplus_{i \geq 0} A_i$

• k -bil. map $A \times A \rightarrow A$, $1 \in A$

"multiplication" $(a, b) \mapsto a \cdot b$ "unit"

s.th. $x(yz) = (xy)z \quad \forall x, y, z \in A$, $1 \cdot a = a = a \cdot 1 \quad \forall a \in A$

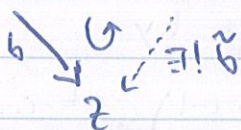
Recall: (Universal property of tensor product)

V, W, Z k -v.s.p

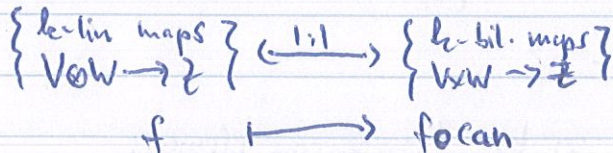
\exists k -v.s.p $V \otimes W$ and k -bil. map $V \times W \xrightarrow{\text{can}} V \otimes W$ s.th. \circ
 $(v, w) \mapsto v \otimes w$

$\forall b: V \times W \rightarrow Z$ k -bil. $\exists!$ $b: V \otimes W \rightarrow Z$ lin.

s.th. $V \times W \xrightarrow{\text{can}} V \otimes W$



In particular:



Def 2 (Rewritten) ass., graded k -alg w/ 1 consists of data

• graded k -v.s.p $A = \bigoplus_{i \geq 0} A_i$

k in degree 0

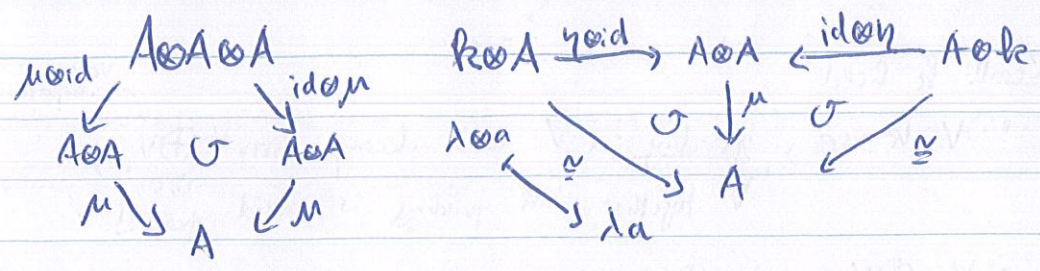
• graded k -lin. maps $A \otimes A \xrightarrow{M} A$ and $\eta: k \rightarrow A$

"multiplication" \uparrow univ. prop.

"unit" \uparrow $1_A \mapsto 1_A$
 \downarrow $1_A := \eta(1_k), 1_A$

$A \times A \xrightarrow{\text{can}} A \otimes A \xrightarrow{M} A, A \times A \rightarrow A$

s.th.



Why this definition?

- (i) Can be used to define algebra objects in arbitrary monoidal categories
- (ii) Definition can be dualized (later today!)

• $F := \mathbb{Z}/2 \langle Sg^i : i \geq 0 \rangle$ "free ass., unital $\mathbb{Z}/2$ -alg. in indeterminates $Sg^i, i \geq 0$ "

$\mathbb{Z}/2$ -vsp. w/ basis finite words $Sg^{i_1} \dots Sg^{i_r}, i_j \in \mathbb{N}_0$
(empty word is unit)

multiplication = concatenation of words (extended bilinearly)

$$(Sg^{i_1} \dots Sg^{i_r}) \cdot (Sg^{j_1} \dots Sg^{j_s}) = Sg^{i_1} \dots Sg^{i_r} Sg^{j_1} \dots Sg^{j_s}$$

Notation: $I = \{i_1, \dots, i_r\}, i_j \in \mathbb{N}_0, Sg^I := Sg^{i_1} \dots Sg^{i_r}$

grading on F given by $d(I) := i_1 + \dots + i_r$

• I (two-sided) ideal in F gen. by

$$Sg^a Sg^b + \sum_{c=0}^{a-1} \binom{b-c-1}{a-2c} Sg^{a+b-c} Sg^c, 0 < a < 2b$$

$$1 + Sg^0$$

Note: I is gen. by homogeneous elements

$\leadsto F/I$ is graded $\mathbb{Z}/2$ -alg. s.th. $p: F \rightarrow F/I$ is a homom. of graded $\mathbb{Z}/2$ -alg.

Def. The ^{associative} graded $\mathbb{Z}/2$ -alg w/ 1 $A := F/I$ is called the (mod-2) Steenrod algebra.

$$m(s_\ell) = \left(\sum_{\substack{s=1 \\ s \neq j, j+1}}^r i_s \cdot s \right) + (i_j + i_{j+1} - \ell) \cdot j + \ell \cdot (j+1)$$

$$\Rightarrow m(I) - m(s_\ell) = j \cdot j + i_{j+1}(j+1) - (i_j + i_{j+1} - \ell)j - \ell(j+1)$$

$$= i_{j+1} - \ell > 0$$

$$i_{j+1} > \frac{i_j}{2} \geq \lfloor \frac{i_j}{2} \rfloor \geq \ell$$

\Rightarrow "..."

\Rightarrow Claim (algorithm terminates because m is bounded) \square

Proof (Thm): \circ spanning set (prop above)

\circ linearly independent: $\left. \begin{array}{l} \text{admissible} \\ \text{admissible} \end{array} \right\}$

$$\lambda_1 s_q^{I_1} + \dots + \lambda_k s_q^{I_k} = 0$$

$$m = \max_{1 \leq i \leq k} \{d(I_i)\}, \quad H^*(\mathcal{K}(\mathbb{Z}/2, m), \mathcal{V}_2)$$

\downarrow
 i_m

Recall (Talk 4): $s_q^{I_i}(i_m)$ lin. indep. in $H^*(\mathcal{K}(\mathbb{Z}/2, m), \mathcal{V}_2)$ $\left. \begin{array}{l} \\ \\ \end{array} \right\} (*)$
for $d(I) \leq m$ and I admissible.

$$0 = \underbrace{(\lambda_1 s_q^{I_1} + \dots + \lambda_k s_q^{I_k})}_{=0} \cdot i_m = \lambda_1 s_q^{I_1}(i_m) + \dots + \lambda_k s_q^{I_k}(i_m)$$

$$\Rightarrow \lambda_1 = \lambda_2 = \dots = \lambda_k = 0. \quad \square$$

(*)

Ex 1

$$s_q^1 s_q^2 s_q^3 = s_q^1 \overset{0 \text{ (Adem)}}{\cancel{s_q^2}} s_q^5 + s_q^1 s_q^4 s_q^1 = s_q^5 s_q^1$$

$\left. \begin{array}{l} \text{1st step of algo:} \\ \text{Apply Adem here} \end{array} \right\}$
 $\left. \begin{array}{l} \text{Apply Adem} \end{array} \right\}$

Moment:

$$1 \cdot 1 + 2 \cdot 2 + 3 \cdot 3 = 14$$

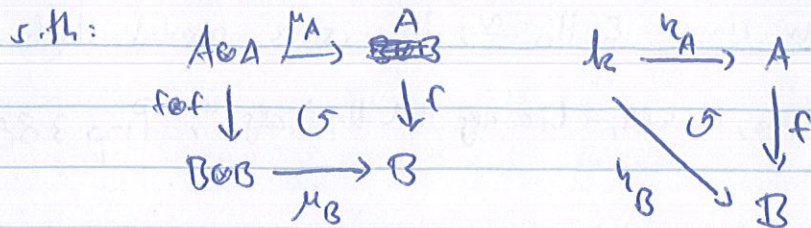
Moment:

$$1 \cdot 1 + 2 \cdot 5 = 11$$

Moment:

$$1 \cdot 1 + 4 \cdot 2 + 1 \cdot 1 = 10$$

• mapism of graded k -alg A, B : $f: A \rightarrow B$ graded k -lin. map



• graded k -vsp $A \otimes B$ becomes graded k -alg by setting:

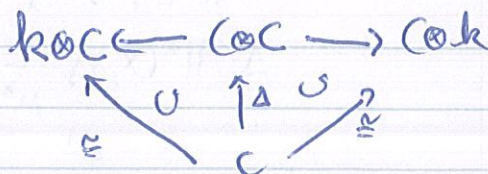
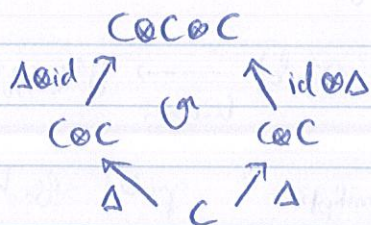
$$\begin{aligned}
 \mu: A \otimes B \otimes A \otimes B &\rightarrow A \otimes B \\
 a \otimes b \otimes \tilde{a} \otimes \tilde{b} &\mapsto (-1)^{|b||\tilde{a}|} (a\tilde{a}) \otimes (b\tilde{b})
 \end{aligned}$$

$$\eta: k \rightarrow A \otimes B \text{ defined by } k \xrightarrow{\cong} k \otimes k \xrightarrow{\eta_A \otimes \eta_B} A \otimes B$$

Def. graded k -coalgebra consists of data:

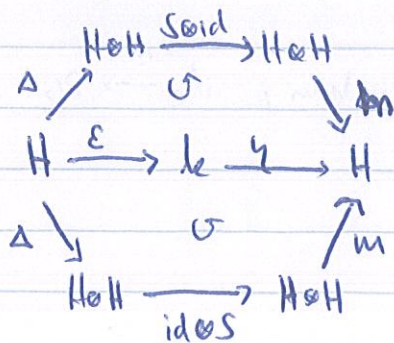
- graded k -vsp C
- graded k -lin. maps $\Delta: C \rightarrow C \otimes C$, $\epsilon: C \rightarrow k$
"comultiplication" "counit"

satisfying:



Def. (i) A graded bialgebra is a graded k -alg which is also a graded k -coalg. s.t. Δ, ϵ are graded k -alg. homom.

(ii) A graded Hopf alg H is a graded bialg together with a k -lin map $S: H \rightarrow H$ s.t. "antipode"



Prop: A graded bialg H w/ property that $\varepsilon: H \rightarrow k$ induces an isom. $\varepsilon: H_0 \xrightarrow{\cong} k$ is a graded Hopf alg.

Proof: "Algebras, rings, modules - Lie alg and Hopf alg.", Prop 3.88 \square

Ex: (1) $\mathbb{C}[X] = \bigoplus_{i \geq 0} \{ \lambda X^i \mid \lambda \in \mathbb{C} \}$ grad \mathbb{C} -vsp.

$$\mu: \mathbb{C}[X] \otimes \mathbb{C}[X] \rightarrow \mathbb{C}[X]$$

$$f \otimes g \mapsto f \cdot g$$

$$\eta: \mathbb{C} \hookrightarrow \mathbb{C}[X]$$

$$\Delta: \mathbb{C}[X] \rightarrow \mathbb{C}[X] \otimes \mathbb{C}[X]$$

$$X^i \mapsto \sum_{k+l=i} \binom{i}{k} X^k \otimes X^l$$

$$\varepsilon: \mathbb{C}[X] \rightarrow \mathbb{C}$$

$$X^i \mapsto \begin{cases} 1 & i=0 \\ 0 & \text{else} \end{cases}$$

isom. in degree 0!

} hom. of algebras!

(2) X top. group, k field

$\leadsto H^*(X, k)$ grad. k -alg.

$$\Delta: H^*(X, k) \xrightarrow{\mu^*} H^*(X \times X, k) \xrightarrow[\text{K\"unneth}]{\cong} H^*(X, k) \otimes H^*(X, k)$$

$\mu: X \times X \rightarrow X$ grp multipl. graded alg. hom!

X path-connected $\Rightarrow H^0(X, k) \cong k \leadsto \varepsilon: H^*(X, k) \rightarrow k$.

Hopf alg. structure of A :

A graded \mathcal{U}_2 -vsp.

• $\mu: A \otimes A \rightarrow A$
 $S_q I \otimes S_q J \rightarrow S_q I S_q J$ (multiplication not commutative!)

$\eta: \mathcal{U}_2 \rightarrow A$

• $\varepsilon: A \rightarrow \mathcal{U}_2$ alg. hom, $A_0 \xrightarrow{\cong} \mathcal{U}_2$
 $S_q^0 \mapsto 1$

$\Delta: A \rightarrow A \otimes A$??