The mod-2 Steenrod algebra

Recall: k field

- V k-vsp, grading on V is a decomposition $V = \bigoplus_{i \geq 0} V_i$, (vector subspaces)
  V together with grading is called graded

- $V = \bigoplus_{i\geq 0} V_i$, $W = \bigoplus_{i \geq 0} W_i$ graded k-vsp.

  (i) k-lin map $f: V \to W$ is graded if $f(V_i) \subseteq W_i$ $\forall i \geq 0$.

  (ii) $V \otimes W$ is equipped w/ grading
  $$(V\otimes W)_i = \text{"vector subspace generated by } v\otimes w \text{ s.th. } |v| + |w| = i\text{"}$$

Def 1 associative, graded k-alg w/ 1 consists of data:

- graded k-vsp $A = \bigoplus_{i\geq 0} A_i$
- k-bil. map $\underset{\text{"multiplication"}}{A \times A \to A}$, $(a,b) \mapsto a\cdot b$, $A_i \cdot A_j \subseteq A_{i+j}$, $\underset{\text{"unit"}}{1 \in A}$

s.th. $x(yz) = (xy)z$ $\forall x,y,z \in A$, $1\cdot a = a = a \cdot 1$ $\forall a \in A$

Recall: (Universal property of tensor product)
V, W, Z k-vsp

$\exists$ k-vsp $V\otimes W$ and k-bil. map $V \times W \xrightarrow{\text{can}} V \otimes W$, $(v,w) \mapsto v\otimes w$ s.th.:

$$\forall b: V\times W \to Z \ k\text{-bil.} \ \exists! \tilde{b}: V \otimes W \to Z \text{ lin.}$$

s.th.
$$\begin{tikzcd} V\times W \arrow[r, "\text{can}"] \arrow[dr, "b"'] & V \otimes W \arrow[d, dashed, "\exists!\tilde{b}"] \\ & Z \end{tikzcd}$$
($\circlearrowleft$)

In particular:
$$\left\{ \begin{matrix} k\text{-lin maps} \\ V\otimes W \to Z \end{matrix} \right\} \xleftrightarrow{1:1} \left\{ \begin{matrix} k\text{-bil. maps} \\ V \times W \to Z \end{matrix}\right\}$$
$$f \longmapsto f \circ \text{can}$$

Def 1 (Rewritten) ass., graded k-alg w/ 1 consists of data

- graded k-vsp $A = \bigoplus_{i \geq 0} A_i$
- graded k-lin. maps $A \otimes A \xrightarrow{\mu} A$ and $\eta: k \to A$ (k in degree 0)
  "multiplication" "unit"

$A\times A \xrightarrow{\text{can}} A \otimes A \xrightarrow{\mu} A$, $A \times A \to A$ (univ. prop.)

$1_A := \eta(1_k)$, 1_A ($1 \mapsto 1_A$)

s.th.

$$\begin{tikzcd}[column sep=small]
& A\otimes A\otimes A \arrow[dl, "\mu\otimes id"'] \arrow[dr, "id\otimes\mu"] & \\
A\otimes A \arrow[dr, "\mu"'] \arrow[rr, phantom, "\circlearrowleft"] & & A\otimes A \arrow[dl, "\mu"] \\
& A &
\end{tikzcd}
\qquad
\begin{tikzcd}
k\otimes A \arrow[r, "\eta\otimes id"] \arrow[dr, "\cong"'] & A\otimes A \arrow[d, "\mu"] & A\otimes k \arrow[l, "id\otimes\eta"'] \arrow[dl, "\cong"] \\
& A &
\end{tikzcd}$$

$\lambda\otimes a \mapsto \lambda a$

<u>Why this definition?</u>

(i) Can be used to define algebra objects in arbitrary monoidal categories

(ii) Definition can be dualized (later today!)

- $F := \mathbb{Z}/2\langle Sq^i : i \geq 0\rangle$ "free ass., unital $\mathbb{Z}/2$-alg. in indeterminates Sq^i, $i\geq 0$"

  $\mathbb{Z}/2$-vsp. w/ basis finite words $Sq^{i_1}\dots Sq^{i_r}$, $i_j\in\mathbb{N}_0$ (empty word is unit)

  multiplication = concatenation of words (extended bilinearly)

  $(Sq^{i_1}\dots Sq^{i_r})\cdot(Sq^{j_1}\dots Sq^{j_s}) = Sq^{i_1}\dots Sq^{i_r}Sq^{j_1}\dots Sq^{j_s}$

  <u>Notation:</u> $I=\{i_1,\dots,i_r\}$, $i_j\in\mathbb{N}_0$, $Sq^I := Sq^{i_1}\dots Sq^{i_r}$

  grading on F given by $d(I) := i_1+\dots+i_r$

- I (two-sided) ideal in F gen. by

$$Sq^a Sq^b + \sum_{c=0}^{[\frac{a}{2}]}\binom{b-c-1}{a-2c} Sq^{a+b-c}Sq^c, \quad 0<a<2b$$

$$1+Sq^0$$

<u>Note:</u> I is gen. by homogeneous elements

$\rightsquigarrow$ F/I is graded $\mathbb{Z}/2$-alg s.th. $p: F \twoheadrightarrow F/I$ is a homom. of graded $\mathbb{Z}/2$-algs.

<u>Def.</u> The (associative) graded $\mathbb{Z}/2$-alg w/ 1 $\mathcal{A} := F/I$ is called <u>the (mod-2) Steenrod algebra.</u>

$$\begin{array}{ccc} F & \xrightarrow{\text{act}} & \mathrm{End}_{\mathbb{Z}/2}(H^*(X;\mathbb{Z}/2)) \\ Sq^I & \longmapsto & Sq^I : H^*(X;\mathbb{Z}/2) \to H^*(X;\mathbb{Z}/2) \end{array}$$

$$p : F \to F/I, \quad Sq^I \mapsto Sq^I, \qquad \exists!\ \overline{\text{act}} : F/I \to \mathrm{End}_{\mathbb{Z}/2}(H^*(X;\mathbb{Z}/2)) \iff I \subseteq \ker(\text{act})$$

$$F/I \overset{\text{"}}{=} \mathcal{A}$$

$$\rightsquigarrow \quad \mathcal{A} \times H^*(X;\mathbb{Z}/2) \longrightarrow H^*(X;\mathbb{Z}/2)$$
$$(Sq^I, x) \longmapsto Sq^I(x) =: Sq^I . x$$

" $H^*(X;\mathbb{Z}/2)$ (graded) $\mathcal{A}$-module "

Def. $I = \{i_1, \dots, i_r\}$, $i_j \in \mathbb{N}_0$, is called <u>admissible</u> if $i_j \geq 2 i_{j+1}$ $\forall j < r$.

Thm (Serre - Cartan)

The monomials Sq^I, I admissible, form a basis of $\mathcal{A}$ as $\mathbb{Z}/2$-vsp.

Ex: $\mathcal{A}_7 = \langle Sq^7, Sq^6 Sq^1, Sq^5 Sq^2, Sq^4 Sq^2 Sq^1 \rangle$

7th graded component

Prop: Any compos. Sq^I of Steenrod sq. is a sum of admissible compositions.

Proof. $Sq^I = Sq^{i_1} \dots Sq^{i_r}$

<u>Algorithm:</u> Read from ~~left~~ right to ~~right~~ left

If $i_j < 2 i_{j+1}$

use Adem-rel. $Sq^{i_1} \dots \underbrace{Sq^{i_j} Sq^{i_{j+1}}}_{= \sum' \dots} \dots Sq^{i_r}$

Repeat for each summand

<u>Claim:</u> This algorithm terminates!

$$m(I) := \sum_{s=1}^{r} i_s s \geq 0 \quad \text{"moment"}$$

"After applying an Adem-rel. every summand of the term has <u>strictly</u> smaller moment than the composition we started with."

Fix ℓ, $0 \leq \ell \leq \lfloor \frac{i_j}{2} \rfloor$, consider the corresp. summand

$$S_\ell := Sq^{i_1} Sq^{i_2} \dots Sq^{i_j + i_{j+1} - \ell} Sq^{\ell} \dots Sq^{i_r}$$

$$m(s_\ell) = \left(\sum_{\substack{s=1 \\ s\neq j, j+1}}^{r} i_s \cdot s\right) + (i_j + i_{j+1} - \ell)\cdot j + \ell\cdot(j+1)$$

$$\Rightarrow \quad m(I) - m(s_\ell) = i_j \cdot j + i_{j+1}(j+1) - (i_j + i_{j+1} - \ell)j - \ell(j+1)$$

$$= i_{j+1} - \ell > 0$$

$$\Updownarrow$$

$$i_{j+1} > \frac{i_j}{2} \geq \left\lfloor \frac{i_j}{2} \right\rfloor \geq \ell$$

$\Rightarrow$ "..."

$\Rightarrow$ Claim (algorithm terminates because m is bounded) $\boxtimes$

Proof (Thm): ∘ spanning set (prop above)

∘ linearly independent: admissible

$$\lambda_1 Sq^{I_1} + \ldots + \lambda_k Sq^{I_k} = 0$$

$$m = \max_{1\leq i\leq k} \{d(I_i)\}, \quad H^*(K(\mathbb{Z}/2, m), \mathbb{Z}/2)$$
$$\ni i_m$$

Recall (Talk 4): $Sq^I(i_m)$ lin. indep. in $H^*(K(\mathbb{Z}/2,m),\mathbb{Z}/2)$ for $d(I)\leq m$ and I admissible. $\Big\}(*)$

$$0 = (\underbrace{\lambda_1 Sq^{I_1} + \ldots + \lambda_k Sq^{I_k}}_{=0}).\, i_m = \lambda_1 Sq^{I_1}(i_m) + \ldots + \lambda_k Sq^{I_k}(i_m)$$

$$\underset{(*)}{\Rightarrow} \quad \lambda_1 = \lambda_2 = \ldots = \lambda_k = 0. \quad \boxtimes$$

Ex:

$$\underbrace{Sq^1 Sq^2}_{\text{1st step of algo: Apply Adem here}} Sq^3 \overset{(\text{Adem})}{=} \cancel{Sq^1 Sq^5}^{\,0} + Sq^1 \underbrace{Sq^4 Sq^1}_{\text{Apply Adem}} = Sq^5 Sq^1$$

moment: $1\cdot1 + 2\cdot2 + 3\cdot3 = 14$

moment: $1\cdot1 + 2\cdot5 = 11$

moment: $1\cdot1 + 4\cdot2 + 1\cdot1 = 10$

- <u>morphism of graded k-alg A, B</u>: $f: A \to B$ graded k-lin. map s.th.:

$$\begin{tikzcd} A\otimes A \arrow[r, "\mu_A"] \arrow[d, "f\otimes f"'] \arrow[dr, phantom, "\circlearrowleft"] & A \arrow[d, "f"] \\ B\otimes B \arrow[r, "\mu_B"'] & B \end{tikzcd} \qquad \begin{tikzcd} k \arrow[r, "\eta_A"] \arrow[dr, "\eta_B"'] & A \arrow[d, "f"] \arrow[dl, phantom, "\circlearrowleft"] \\ & B \end{tikzcd}$$

- graded k-vsp $\underbrace{A\otimes B}_{\text{graded } k\text{-alg.}}$ becomes graded k-alg by setting:

$$\mu: A\otimes B \otimes A\otimes B \longrightarrow A\otimes B$$
$$a\otimes b\otimes \tilde{a}\otimes\tilde{b} \longmapsto (-1)^{|b||\tilde{a}|}(a\tilde{a})\otimes(b\tilde{b})$$

$$\eta: k \longrightarrow A\otimes B \text{ defined by } k \xrightarrow{\cong} k\otimes k \xrightarrow[\eta_A\otimes\eta_B]{} A\otimes B$$

<u>Def.</u> <u>graded k-coalgebra</u> consists of data:

- graded k-vsp C
- graded k-lin. maps $\Delta: C \to C\otimes C$, $\varepsilon: C \to k$
  "comultiplication" "counit"

satisfying:

$$\begin{tikzcd} & C\otimes C\otimes C & \\ C\otimes C \arrow[ur, "\Delta\otimes id"] & \circlearrowleft & C\otimes C \arrow[ul, "id\otimes\Delta"'] \\ & C \arrow[ul, "\Delta"] \arrow[ur, "\Delta"'] & \end{tikzcd} \qquad \begin{tikzcd} k\otimes C & C\otimes C \arrow[l] \arrow[r] & C\otimes k \\ & C \arrow[ul, "\varepsilon"] \arrow[u, "\Delta"] \arrow[ur, "\cong"'] & \end{tikzcd}$$

<u>Def.</u> (i) A <u>graded bialgebra</u> is a graded k-alg which is also a graded k-coalg. s.th. Δ, ε are graded k-alg. homom.

(ii) A <u>graded Hopf alg</u> H is a graded bialg together with a k-lin map $S: H \to H$ s.th.
"antipode"

$$\begin{tikzcd} & H\otimes H \arrow[r, "S\otimes id"] & H\otimes H \arrow[dr, "m"] & \\ H \arrow[ur, "\Delta"] \arrow[rr, phantom, "\circlearrowleft"] \arrow[r, "\varepsilon"] \arrow[dr, "\Delta"'] & k \arrow[rr, "\eta"] & & H \\ & H\otimes H \arrow[r, "id\otimes S"'] & H\otimes H \arrow[ur, "m"'] & \end{tikzcd}$$

Prop: A graded bialg H w/ property that $\varepsilon: H \to k$ induces an isom. $\varepsilon: H_0 \xrightarrow{\sim} k$ is a graded Hopf alg.

Proof: "Algebras, rings, modules – Lie alg and Hopf alg.", Prop 3.88 $\blacksquare$

Ex: (1) $\mathbb{C}[X] = \bigoplus_{i \geq 0} \{ \lambda X^i \mid \lambda \in \mathbb{C} \}$ grad $\mathbb{C}$-vsp.

$$\mu: \mathbb{C}[X] \otimes \mathbb{C}[X] \longrightarrow \mathbb{C}[X]$$
$$f \otimes g \longmapsto f \cdot g$$

$$\eta: \mathbb{C} \hookrightarrow \mathbb{C}[X]$$

$$\Delta: \mathbb{C}[X] \longrightarrow \mathbb{C}[X] \otimes \mathbb{C}[X]$$
$$X^i \longmapsto \sum_{k+l=i} \binom{i}{k} X^k \otimes X^l$$

$$\varepsilon: \mathbb{C}[X] \longrightarrow \mathbb{C}$$
$$X^i \longmapsto \begin{cases} 1 & i=0 \\ 0 & \text{else} \end{cases}$$

isom. in degree 0!

hom. of algebras!

(2) X top. group, k field

$\rightsquigarrow H^*(X,k)$ grad. k-alg.

$$\Delta: H^*(X,k) \xrightarrow[\mu^*]{} H^*(X \times X, k) \xrightarrow[\text{Künneth}]{\cong} H^*(X,k) \otimes H^*(X,k)$$

$\mu: X \times X \to X$ grp multipl. graded alg. hom!

X path-connected $\Rightarrow H^0(X,k) \cong k \rightsquigarrow \varepsilon: H^*(X,k) \to k$.

Hopf alg. structure of $\mathcal{A}$:

$\mathcal{A}$ graded $\mathbb{Z}/2$-vsp.

- $\mu: \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$
  $Sq^I \otimes Sq^J \to Sq^I Sq^J$
  (multiplication not commutative!)

  $\eta: \mathbb{Z}/2 \to \mathcal{A}$
- $\varepsilon: \mathcal{A} \to \mathbb{Z}/2$ alg. hom, $\mathcal{A}_0 \xrightarrow{\cong} \mathbb{Z}/2$
  $Sq^0 \mapsto 1$

  $\Delta: \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$??