Talk 4 Properties of the squares.

Theorem 1. The operation Sq^i, defined (for $i \geq 0$) in the previous talk, have the following properties:

0. Sq^i is a natural homomorphism
1. If $i > p$, $Sq^i(x) = 0$ for $|x| = p$.
2. $Sq^i(x) = x^2$ for $|x| = i$
3. Sq^0 is the identity homomorphism.
4. Sq^1 is the Bockstein homomorphism.
5. $\delta^* Sq^i = Sq^i \delta^*$, where $\delta^* : H^*(L; \mathbb{Z}/2) \to H^*(K, L; \mathbb{Z}/2)$
6. Cartan formula: $Sq^i(xy) = \sum_j (Sq^j x)(Sq^{i-j} y)$.
7. Adem relations: For $a < 2b$, $Sq^a Sq^b = \sum_c \binom{b-c-1}{a-2c} Sq^{a+b-c} Sq^c$.
   where the binomial coefficient is taken mod 2.

Remark: The above properties completely characterize the squaring operations and may be taken as axioms.

(0), (1), (2), (5) have been proved in Talk 3.

Talk 4 will be devoted to the proof of (3), (4), (6).

Some preparations for the proof of (7) are also given.

1. Sq^1 and Sq^0.

Recall that:

$\beta : H^*(K, L; \mathbb{Z}/2) \to H^{*+1}(K, L; \mathbb{Z})$ the Bockstein homomorphism of the exact coefficient sequence $0 \to \mathbb{Z} \xrightarrow{\times 2} \mathbb{Z} \to \mathbb{Z}/2 \to 0$.

$\delta_2 : H^*(K, L; \mathbb{Z}/2) \to H^{*+1}(K, L; \mathbb{Z}/2)$ the Bockstein homomorphism of the sequence $0 \to \mathbb{Z}/2 \xrightarrow{\times 2} \mathbb{Z}/4 \to \mathbb{Z}/2 \to 0$

β is defined as:

$$\begin{array}{ccc} \mathbb{Z} & \mathbb{Z} & \mathbb{Z}/2 \\ & c' & \mapsto c \\ & \downarrow & \\ \frac{1}{2}\delta c' & \mapsto \delta c' & \end{array}$$

$\beta(\bar{c}) = \overline{\frac{1}{2}\delta c'}$.

Recall that we have

$\delta_2 = \rho_2 \beta$

where $\rho_2 : H^*(K, L; \mathbb{Z}) \to H^*(K, L; \mathbb{Z}/2)$ is the mod 2 reduction.

Lemma 1. $\delta_2 Sq^j = \begin{cases} 0 & j \text{ odd} \\ Sq^{j+1} & j \text{ even} \end{cases}$

Remark. Lemma 1 $\Longrightarrow$ (4) follows from (3)

i.e. Sq^0 is identity $\Rightarrow$ Sq^1 is Bockstein.

Proof of (3): Hint, $\mathbb{R}P^2$ ✓ $H^*(\mathbb{R}P^2; \mathbb{Z}/2) = \mathbb{Z}/2[\alpha]/\langle\alpha^3\rangle$.

S^1: $H^1(S^1; \mathbb{Z}/2) \cong \mathbb{Z}/2\,\sigma$, $f: S^1 \to \mathbb{R}P^2$. $f^*\alpha = \sigma$ ✓

S^n: ✓ $H^n(S^n; \mathbb{Z}/2) = \mathbb{Z}/2\,\sigma$.

Exercise. K: $\dim K = n$. $\forall x \in H^n(K; \mathbb{Z}/2)$

(P.7) Hopf-Whitney theorem $\Rightarrow \exists f: K \to S^n$ s.t. $f^*\sigma = x$.

K: unrestricted dimension, $i: K^n \hookrightarrow K$

$i^*: H^n(K; \mathbb{Z}/2) \to H^n(K^n; \mathbb{Z}/2)$ injective.

(K,L): $H^*(K,L) \cong H^*(K \cup CL, CL) \cong \tilde{H}^*(K \cup CL)$.

$\blacksquare$

Proof of Lemma 1.: $u \in H^p(K,L;\mathbb{Z}/2)$, $u = \overline{c'}$

$$\begin{array}{c} c \longmapsto c' \\ \downarrow \\ a \xmapsto{\times 2} \delta c = 2a \end{array}$$

Hence. $\beta u = \overline{a} \Longrightarrow \delta_2 u = \overline{a} \mod 2$.

$Sq^j u = \overline{c' \cup_{p-j} c'} = \overline{c \cup_{p-j} c} \mod 2$, $\delta_2 Sq^j u = ?$

Let $i = p-j$.

Coboundary formula $\Longrightarrow$ $\delta(c \cup_i c) = (-1)^i \delta c \cup_i c + (-1)^j c \cup_i \delta c$

$- (-1)^i c \cup_{i-1} c - (-1)^p c \cup_{i-1} c$

$= (-1)^i 2a \cup_i c + (-1)^j c \cup_i 2a$

$- (-1)^i c \cup_{i-1} c - (-1)^p c \cup_{i-1} c$

$\Rightarrow \beta(\overline{c' \cup_i c'}) = \begin{cases} \overline{(-1)^i a \cup_i c + (-1)^j c \cup_i a} & j \text{ odd} \\ \overline{(-1)^i a \cup_i c + (-1)^j c \cup_i a - (-1)^p c \cup_{i-1} c} & j \text{ even} \end{cases}$

$\Rightarrow \delta_2 Sq^j u = \delta_2(\overline{c' \cup_i c'}) = \cancel{\beta_2 \delta_2}\, \rho_2 \beta(\overline{c' \cup_i c'})$

remark c) of Thm 2 $\Rightarrow$ $= \begin{cases} 0 \\ \overline{c \cup_{i-1} c} \mod 2 \end{cases} \overset{\#}{=} \begin{cases} 0 \\ \overline{c \cup_{i-1} c} \mod 2 \end{cases} = \begin{cases} 0 & j \text{ odd} \\ Sq^{j+1} u & j \text{ even} \end{cases}$ $\blacksquare$

2. Cartan Formula.

$$\varphi_{K\otimes L}: \quad W\otimes K\otimes L \xrightarrow{r\otimes 1} W\otimes W\otimes K\otimes L \xrightarrow{T} W\otimes K\otimes W\otimes L$$

$$\xrightarrow{\varphi_K\otimes\varphi_L} K\otimes K\otimes L\otimes L \xrightarrow{T} K\otimes L\otimes K\otimes L.$$

where T permutes the second and third factors.

$\dim u = p$, $\dim v = q$, $n = p+q-i$

$$\begin{aligned} Sq^i(u\times v)(a\otimes b) &= \big((u\otimes v)\cup_n (u\otimes v)\big)(a\otimes b) \\ &= (u\otimes v\otimes u\otimes v)\,\varphi_{K\otimes L}(d_n\otimes a\otimes b) \\ &= (u\otimes v\otimes u\otimes v)\,T\cdot\varphi_K\otimes\varphi_L\,T\cdot r\otimes 1\,(d_n\otimes a\otimes b) \\ &= (u\otimes u\otimes v\otimes v)\sum_{j=0}^{n}\varphi_K(d_j\otimes a)\otimes T^j\varphi_L(d_{n-j}\otimes b) \\ &= \sum_{j=0}^{n}(u\cup_j u)(a)\otimes(v\cup_{n-j} v)(b) \\ &= \sum_{j=0}^{n}(Sq^{p-j}u\times Sq^{q-n+j}v)(a\otimes b) \end{aligned}$$

$$\Rightarrow \quad Sq^i(u\times v) = \sum_{j=0}^{n} Sq^{p-j}u\times Sq^{q-n+j}v$$

$$= \sum_{s=i-q}^{p} Sq^s u\times Sq^{i-s}v \qquad s = p-j$$

$$= \sum_{s=0}^{i} Sq^s u\times Sq^{i-s}v$$

$\Rightarrow$ Note that we have $x\cup y = \Delta^*(x\times y)$ $\quad\Delta$: diagonal map

$$Sq^i(x\cup y) = \Delta^* Sq^i(x\times y) = \sum Sq^j x\cup Sq^{i-j}y. \qquad \boxed{}$$

$\Longrightarrow$ Sq^i group homomorphism not ring homomorphism.

$Sq = \sum Sq^i$ ring homomorphism.

For $u\in H^1(K;\mathbb{Z}_2)$. $\quad Sq^i(u^j) = \binom{j}{i}u^{j+i}$.

3. Squares in $\prod_n K(\mathbb{Z}/2, 1)$.

Denote by $K_n = \prod_{i=1}^{n} K^i$, $K^i = K(\mathbb{Z}/2, 1)$, for each i

$$H^*(K^i; \mathbb{Z}/2) \cong \mathbb{Z}/2[x_i] \implies H^*(K_n; \mathbb{Z}/2) = \mathbb{Z}/2[x_1, x_2 \cdots x_n].$$

Let. $\sigma_1 = x_1 + x_2 + \cdots + x_n$, $\sigma_2 = \sum_{1\le i<j\le n} x_i x_j$, $\cdots$, $\sigma_n = x_1 x_2 \cdots x_n$ be the elementary symmetric function of $x_1, x_2 \cdots x_n$.

Since $H^n(K_n; \mathbb{Z}/2) \cong [K_n, K(\mathbb{Z}/2, n)]$, $\sigma_n \in H^n(K_n; \mathbb{Z}/2)$

$\rightsquigarrow \exists$. $f: K_n \to K(\mathbb{Z}/2, n)$. s.t. $f^* \iota_n = \sigma_n$

where ι_n is the fundamental class of $K(\mathbb{Z}/2, n)$

☆ What we want to do is to prove:

$$f^*: H^*(K(\mathbb{Z}/2, n); \mathbb{Z}/2) \longrightarrow H^*(K_n; \mathbb{Z}/2)$$

is injective for $* \le 2n$.

3.1. Cohomology of $H^*(K(\mathbb{Z}/2, n); \mathbb{Z}/2)$

Notations: Sequence $I = \{i_1, i_2, \cdots, i_r\}$. with $i_r > 0$ i_r integer.

$Sq^I \triangleq Sq^{i_1} Sq^{i_2} \cdots Sq^{i_r}$. ; if $I = \phi$, $Sq^I \triangleq Sq^0$.

Example: $Sq^{(4,2,1)} = Sq^4 Sq^2 Sq^1$.

Def: I is admissible if $i_j \ge 2 i_{j+1}$; if I is admissible, we say that Sq^I is admissible

Example: $I = (4, 2, 1)$, $I = (5, 2, 1)$, $\cdots$.

$I = (i)$ $i > 0$.

Def: $I = \{i_1, i_2, \cdots, i_r\}$

$l(I) = r$. length of I;

$d(I) = \sum_{j=1}^{r} i_j$. the degree of I; $Sq^I: H^*(X; \mathbb{Z}/2) \to H^{*+d(I)}(X; \mathbb{Z}/2)$

If I is admissible,

$$e(I) = 2i_1 - d(I) = 2i_1 - i_1 - i_2 - \cdots - i_r$$
$$= (i_1 - 2i_2) + (i_2 - 2i_3) + \cdots + (i_{r-1} - 2i_r) + i_r$$

the excess of I.

✻ Serre's theorem:

$$H^*(K(\mathbb{Z}/2, n); \mathbb{Z}/2) \cong \mathbb{Z}/2[Sq^I(\iota_n)],$$ I runs through all admissible sequence I with $e(I) < n$.

Remark. ① Serre's theorem can not be proved here.

② (Proposition 4). K. any CW-complex, $\forall u \in H^n(K; \mathbb{Z}/2)$.
if $e(I) > n$, $Sq^I(u) = 0$.
$e(I) = n$, $Sq^I(u) = (Sq^J(u))^2$, where if $I = (i_1, i_2, \dots, i_r)$, $J = (i_2, \dots, i_r)$.

Proof: Exercise.

③ (P_{28}, Corollary of Theorem 2). $Sq^I(\iota_n)$ are linearly independent.
Proof: After Theorem 2. Exercise.

④ (Corollary of Proposition 3) $Sq^i \iota_n \neq 0$. for $0 < i \leq n$.

3.2 Steenrod Squares on the cohomology of K_n

Proposition 3. $Sq^i \sigma_n = \sigma_n \sigma_i$. $1 \leq i \leq n$.

Proof. $Sq\, \sigma_n = Sq\left(\prod_{i=1}^{n} x_i\right) = \prod_{i=1}^{n} Sq\, x_i = \cdots = \sigma_n \cdot \left(\sum_{i=0}^{n} \sigma_i\right)$

$\Rightarrow Sq^i \sigma_n = \sigma_n \sigma_i$.

Remark: Remark ④ above is true.

$S = \mathbb{Z}/2[\sigma_1, \cdots, \sigma_n] \subset \mathbb{Z}/2[x_1, \cdots x_n] \cong H^*(K_n; \mathbb{Z}/2)$.

Ordering on the monomials of S:
For any monomial $m = \sigma_{j_1}^{e_1} \sigma_{j_2}^{e_2} \cdots \sigma_{j_s}^{e_s}$. with $j_1 > j_2 > \cdots > j_s$

Def: $m < m'$ if $j_1 < j_1'$ or
if $j_1 = j_1'$ and $m/\sigma_{j_1} < m'/\sigma_{j_1'}$

Example 1. $\sigma_3 \sigma_2 < \sigma_3^2 \cdot \sigma_1$

Example 2. Let x be an monomials with $\deg x = i+j$. $x \in S$.
then. $x \le \sigma_{i+j}$.
$\Rightarrow Sq^j \sigma_i \le \sigma_{i+j}$. $Sq^i \sigma_i = \sigma_i^2 < \sigma_{2i}$

Example 3. $Sq^k m \le \sigma_{2j_1-1} \cdots$.

Theorem 2. I admissible, $e(I) \le n$. $\Rightarrow Sq^I \sigma_n = \sigma_n \cdot Q_I$
where $Q_I = \sigma_{i_1} \cdot \sigma_{i_2} \cdots \sigma_{i_r} + X_I$
X_I is a sum of monomials of low order

Proof: Induction on the lengths of I

$l(I) = 1$. $Sq^i \sigma_n = \sigma_n \sigma_i$. , $Q_I = \sigma_i$. True.

Assume that the facts of Theoren 2 is true for ~~...~~ lengths $< r$.
Let $I = (i_1, i_2, \cdots, i_r)$ $J = (i_2, \cdots, i_r)$ admissible, $l(J) = r-1 < r$

$$\begin{aligned} Sq^I \sigma_n &= Sq^{i_1} Sq^J \sigma_n = Sq^{i_1}(\sigma_n \cdot Q_J) = \sum_{m=0}^{i_1} Sq^m \sigma_n \cdot Sq^{i_1-m} Q_J \\ &= \sigma_n \cdot \sigma_{i_1} \cdot Q_J + \sum_{m=0}^{i_1-1} Sq^m \sigma_n \cdot Sq^{i_1-m} Q_J. \\ &= \sigma_n \cdot \sigma_{i_1} \cdots \sigma_{i_r} + \sigma_n \cdot \sigma_{i_1} \cdot X_J + \sum_{m=0}^{i_1-1} \sigma_n \cdot \sigma_m \cdot Sq^{i_1-m} Q_J \end{aligned}$$

Example 3 $\Rightarrow Sq^{i_1-m} Q_J \le \sigma_{2i_2-1} \cdots$ $\quad 2i_2 - 1 < 2i_2 \le i_1$
$\Rightarrow \sigma_{i_1} \cdot X_J + \sum_{m=0}^{i_1-1} \sigma_m Sq^{i_1-m} Q_J \le \sigma_{i_1} \cdot \sigma_{i_2} \cdots \sigma_{i_r}$. ☒

<u>Remark:</u> ① σ_I, I runs through all admissible sequence of degree $\leq n$, are linearly independent in S, hence in $H^*(K_n; \mathbb{Z}/2)$

Theorem 2 $\Rightarrow$ $Sq^2\sigma_n$ are also linearly independent.

② Remark ③ in 3.1 above is proved.

③ Serre's Theorem + $Sq^2\sigma_n$ linearly independent $\Rightarrow$

$$f^*: H^*(K(\mathbb{Z}/2, n); \mathbb{Z}/2) \longrightarrow H^*(K_n; \mathbb{Z}/2)$$

is injective for $* \leq 2n$, where $f^* e_n = \sigma_n$.