Talk 2. Construction of the $\cup_i$-products

1. The complex $K(\mathbb{Z}/2, 1)$.

An Eilenberg-MacLane space $K(\pi, n)$ is determined up to homotopy by the property

$$\pi_i(K(\pi, n), *) = \begin{cases} 0, & i \neq n \\ \pi, & i = n. \end{cases}$$

It is an H-space (i.e., has a product operation unital and associative up to homotopy, with $*$ the usual base point.

1.1. Proposition $\mathbb{RP}^\infty$ is of homotopy type $K(\mathbb{Z}/2, 1)$, i.e, $\pi_1(\mathbb{RP}^\infty) \cong \mathbb{Z}/2$, $\pi_i(\mathbb{RP}^\infty) = 0$ for $i \neq 1$.

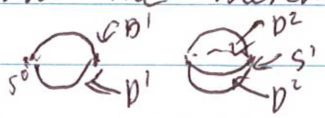
Proof: $\mathbb{RP}^\infty$ is the quotient space of S^∞ modulo the $\mathbb{Z}/2$-action of taking antipodal points, where $S^\infty := \bigcup_{n=0}^\infty S^n$ is contractible. Since the action is free, the projection $S^\infty \to \mathbb{RP}^\infty$ is ~~a~~ universal cover, and the proposition follows. ∎

Remark: A "nice" space has the same π_i, $i > 1$ as its universal cover.

1.2. The cell structures of S^∞ and $\mathbb{RP}^\infty$.

(1) S^∞: We build a cell structure for S^i for each $i \geq 0$ inductively and take the direct limit,

- S^0 is the discrete set of 2 points.
- Attach 2 copies of D^i as "north" and "south" hemisphere along the equator S^{i-1}, via the identity and the antipodal map of S^{n-1}, respectively.



- Let T be the antipodal action, then we've obtained 2 cells d_i and Td_i in each dimension i, and the boundary operator given by

$$\begin{cases} \partial d_i = d_{i-1} + (-1)^i T d_{i-1} \\ \partial T d_i = T d_{i-1} + (-1)^i d_{i-1}. \end{cases}$$

We denote this chain complex by W.

Exercise: Show that W is

i) $\mathbb{Z}/2$-equivariant, i.e. the boundary operator commutes with the $\mathbb{Z}/2$-action; and

ii) acyclic; i.e. $H_i(W)=0$ for $i>0$.

2) $\mathbb{RP}^\infty$: We take the orbits of the $\mathbb{Z}/2$-action on W and obtain the cellular chain complex $C_*(\mathbb{RP}^\infty)$ with one cell e_i in each dimension i and

$$\partial e_i = \begin{cases} 2e_{i-1}, & i \text{ even,} \\ 0 & i \text{ odd.} \end{cases}$$

Proposition 1.3. $H_i(\mathbb{RP}^\infty;\mathbb{Z}/2) \cong H^i(\mathbb{RP}^\infty;\mathbb{Z}/2) \cong \mathbb{Z}/2, \ \forall i \geq 0.$

Proof: Exercise. $\boxtimes$

Proposition 1.4. $H^*(\mathbb{RP}^\infty;\mathbb{Z}/2) \cong \mathbb{Z}/2[\alpha_1]$, where α_1 is of degree 1.

Idea: Recall that a cochain is a linear function on the chain complex, and we therefore have a pairing $\langle -,- \rangle$ of a chain and a cochain. For a space X, let $\Delta: X \to X \times X$ be the diagonal. Recall that by definition, the cup product $\alpha \cup \beta$ is characterized by

$$\langle \alpha \cup \beta, c \rangle = \langle \alpha \times \beta, \Delta_*(c) \rangle, \quad \text{for any } c \in \overset{H_*(X;R).}{\cancel{H_*(X;R)}}$$

Here $\alpha \times \beta \in H^*(X \times X; R)$ is the cross-product.

Hence we seek to find an explicit formula for Δ_*. However, $\Delta: \mathbb{RP}^\infty \to \mathbb{RP}^\infty \times \mathbb{RP}^\infty$ is not a cellular map with respect to our cell complex structure on $\mathbb{RP}^\infty$. However, its universal cover S^∞ is contractible and $\Delta: S^\infty \to S^\infty \times S^\infty$ is $\mathbb{Z}/2$-equivariant, making $\Delta_\#$ more accessible up to homotopy.

Proof of Proposition 1.4: Consider the following diagram.

$$\begin{tikzcd} S^\infty \arrow[r, "\Delta"] \arrow[d] & S^\infty \times S^\infty \arrow[d] \\ \mathbb{R}P^\infty \arrow[r, "\Delta"] & \mathbb{R}P^\infty \times \mathbb{R}P^\infty \end{tikzcd}$$

Since S^∞, and $S^\infty \times S^\infty$ are contractible, and Δ is $\mathbb{Z}/2$-equivarant. $\Delta_\#$ is a $\mathbb{Z}/2$-equivarant chain map between acyclic chain complexes : $\Delta_\# : W \to W \otimes W$. i.e., a $\mathbb{Z}[\pi]$-module chain map. where $\pi = \mathbb{Z}/2$.

Lemma 1.5 : Let A_*, B_* be chain complexes of R-module, where R is a commutative unital ring. Let A_* be free and B_* be acyclic. Suppose that $f, g : A_* \to B_*$ are chain maps inducing the same homomorphism on H_0, then $f \simeq g$ (chain homotopy.

Proof of Lemma 1.5 : We construct the homotopy $h_i : A_i \to B_{i+1}$ inductively.

$i=0$: Diagram chasing :

A_0 free, B acyclic.

$$\begin{tikzcd} & A_0 \arrow[r] \arrow[d] \arrow[dr, "0"] & H_0(A_*) \arrow[r] \arrow[d, "H_0(f) - H_0(g) = 0"] & 0 \\ B_1 \arrow[r] & B_0 \arrow[r] & H_0(B_*) \arrow[r] & 0 \end{tikzcd}$$

$i > 0$: Exercise. ∎

Applying Lemma 1.5, we see that any chain map ($\mathbb{Z}/2$-equivariant) $r : W \to W \otimes W$ s.t. $H_0(r)$ is an isomorphism is homotopic to $\Delta_\#$.

Exercise : Check that the following r satisfies the above :

$$r(d_i) = \sum_{0 \le j \le i} (-1)^{j(i-j)} d_j \otimes T^i d_{i-j}$$

$$r(Td_i) = T(r(d_i))$$

The chain map r induces $s: C_*(\mathbb{R}P^\infty) \otimes C_*(\mathbb{R}P^\infty)$

$$e_i \mapsto \sum_{0 \le j \le i} (-1)^{j(i-j)} e_j \otimes e_{j-i}$$

Let α_i denote the (unique) generator of $H^i(\mathbb{R}P^\infty; \mathbb{Z}/2)$, and $[e_i]$ the homology class of e_i in $H_i(\mathbb{R}P^\infty; \mathbb{Z}/2)$. Then α_i is the dual of $[e_i]$ (Exercise).

$$\langle \alpha_i \cup \alpha_j, e_{i+j} \rangle = \langle \Delta^*(\alpha_i \times \alpha_j), e_{i+j} \rangle \equiv 1 \pmod 2.$$

~~Ex~~Exercise.

$\Rightarrow \alpha_i \cup \alpha_j = \alpha_{i+j}$ ∎

2. $\cup_i$-products.

2.1 Motivation: X-space. R-commutative, unital ring. Cup products are graded-commutative:

$\alpha \in H^i(X;R)$, $\beta \in H^j(X;R)$ $\alpha \cup \beta = (-1)^{i+j} \beta \cup \alpha$.

There should be a homotopy

$$C^i(X;R) \otimes C^j(X;R) \longrightarrow C^{i+j-1}(X;R)$$

and "homotopy between homotopies"

$$C^i(X;R) \otimes C^j(X;R) \longrightarrow C^{i+j-2}(X;R).$$

$$\vdots$$

$$C^i(X;R) \otimes C^j(X;R) \to C^{i+j-n}(X;R).$$

2.2. Let K be the chain complex of a space X. Define $\mathbb{Z}/2$-actions on $W \otimes K$ and $K \otimes K$ by

$$\begin{cases} T(w \otimes x) = T(w) \otimes x \\ T(x \otimes y) = y \otimes x \end{cases}$$

2.3. Lemma: There is a $\mathbb{Z}/2$-equivariant chain map.

$\varphi: W \otimes K \to K \otimes K$. ~~H~~ $H_0(\varphi)$ is an isomorphism

Moreover, $H_0(\varphi)$ determines φ up to homotopy.

Proof: Let $X = \Delta^n$ be the standard n-complex. Consider the following diagram

$$\begin{array}{ccccc} (W\otimes K)_k & \to & (W\otimes K)_{k-1} & \to & (W\otimes K)_{k-2} \\ \downarrow & & \downarrow & & \downarrow \\ (K\otimes K)_k & \to & (K\otimes K)_{k-1} & \to & (K\otimes K)_{k-2} \end{array}$$

Exercise: Inductively construct φ_k, using the fact that $W\otimes K$ is free, and $K\otimes K$ is acyclic.

For a general X, let $\alpha: \Delta^k \to X$ be a singular k-complex. Let

$$\begin{cases} \varphi(d_i \otimes \alpha) = \alpha_\# (d_i \otimes 1_{\Delta^k}) \\ \varphi(Td_i \otimes \alpha) = \alpha_\# (Td_i \otimes 1_{\Delta^k}) \end{cases}$$

Exercise: i) Show that φ is a well-defined $\mathbb{Z}/2$-equivariant chain map such that $H_0\varphi$ is an isomorphism, and that φ is functorial on X.

ii). Use the same idea to construct a homotopy between any two choices of φ, having the same $H_0(\varphi)$.

iii). Google "acyclic model" and "Eilenberg-Zilber Theorem". ∎

2.4 Definition: For each $i \geq 0$, define a "$\cup_i$-product"

$$C^p(K) \otimes C^q(K) \to C^{p+q-i}(K)$$
$$u \otimes v \mapsto u \cup_i v.$$

by the formula

$$(u \cup_i v)(c) = (u\otimes v)\,\varphi(d_i \otimes c), \quad c \in C_{p+q-i}(K)$$

2.5. Coboundary formula.

$$\delta(u \cup_i v) = (-1)^i \delta u \cup_i v + (-1)^{i+p} u \cup_i \delta v - (-1)^i u \cup_{i-1} v$$
$$- (-1)^{pq} u \cup_{i-1} v,$$

where $\cup_{-1}$ is understood to be 0.

Proof: Let $c \in K_{p+q-i+1}$, then.

$$\delta(u \cup_i v)(c) = (u \cup_i v)(\delta c)$$
$$= (u \otimes v)\, \varphi(d_i \otimes c)$$
$$= (u \otimes v)\, \varphi[(-1)^i(\partial(d_i \otimes c) - \partial d_i \otimes c)]$$
$$= (-1)^i [\delta(u \otimes v)(d_i \otimes c) - (u \otimes v)(d_{i-1} \otimes c + (-1)^i T d_{i-1} \otimes c)]$$
$$= (-1)^i [\delta u \cup_i v(c) + (-1)^p u \cup_i \delta v(c) - (u \cup_{i-1} v(c) + (-1)^i (-1)^{pq} v \cup_{i-1} u(c))]$$
$$= (-1)^i \delta u \cup_i v(c) + (-1)^{i+p} u \cup_i \delta v(c) - (-1)^i u \cup_{i-1} v(c) - (-1)^{pq} v \cup_{i-1} u(c). \blacksquare$$

Remark: The $\cup_i$-products above do NOT pass to cohomology in general.

Exercise: Show that $\cup_i$ is functorial.