Talk 3. Construction of Sq^i

1. Construction of $Sq^i : H^p(K,L;\mathbb{Z}/2) \to H^{p+i}(K,L;\mathbb{Z}/2)$. $L \subset K$.

1.1. Remarks on $\cup_i$ and $\delta(u\cup_i v)$.

$(u\cup_i v)(c) = (u\otimes v)\varphi(d_i\otimes c)$ $\qquad$ $\mathbb{Z}$ - coefficient.

$\delta(u\cup_i v) = (-1)^i \delta u \cup_i v + (-1)^{i+p} u \cup_i \delta v$
$\quad -(-1)^i u \cup_{i-1} v - (-1)^{pq} v \cup_{i-1} u$

Remarks: with $\mathbb{Z}/2$ - coefficients, we have

a). $\delta(u\cup_i u) = \delta u \cup_i u + u\cup_i \delta u$

in particular, $\delta u = 0 \Rightarrow \delta(u\cup_i u) = 0$.

b). $(\delta u)\cup_i(\delta u) = \delta(u\cup_i \delta u + u\cup_{i-1} u)$

c). $\delta u = \delta v = 0 \Rightarrow \delta(u\cup_i v) = u\cup_{i-1} v + v\cup_{i-1} u$.

proof: Exercise. ∎

1.2 Construction of $Sq^i : H^p(K;\mathbb{Z}/2) \to H^{p+i}(K;\mathbb{Z}/2)$

Recall that $H^p(K;\mathbb{Z}/2) \triangleq Z^p(K;\mathbb{Z}/2) / B^p(K;\mathbb{Z}/2)$

↳ cocycle group $\qquad$ ↳ coboundary group

step 1. $Sq_i : Z^p(K;\mathbb{Z}/2) \to Z^{2p-i}(K;\mathbb{Z}/2)$

$u \longmapsto u\cup_i u$

① make sense by Remark a).

② $(u+v)\cup_i(u+v) = u\cup_i u + v\cup_i v + \boxed{u\cup_i v + v\cup_i u}$

$\Rightarrow Sq_i$ is not a homomorphism.

step 2. ③ Remark c) $\Rightarrow u\cup_i v + v\cup_i u = \delta(u\cup_{i+1} v)$

② + ③ $\Rightarrow$

$Sq_i : Z^p(K;\mathbb{Z}/2) \to H^{2p-i}(K;\mathbb{Z}/2)$

$u \longmapsto \overline{u\cup_i u}$ (cohomology class)

is a homomorphism.

Step 3: Remark b) $\Rightarrow Sq_i B^q(K;\mathbb{Z}/2) = 0$

Then we have a homomorphism:

$$Sq_i : H^p(K;\mathbb{Z}/2) \longrightarrow H^{2p-i}(K;\mathbb{Z}/2)$$

Step 4: Def: $Sq^i = Sq_{p-i} : H^p(K;\mathbb{Z}/2) \longrightarrow H^{p+i}(K;\mathbb{Z}/2)$

$$0 \le i \le p.$$

1.3. Construction of $Sq^i : H^p(K,L;\mathbb{Z}/2) \longrightarrow H^{p+i}(K,L;\mathbb{Z}/2)$, $L \subset K$.

① $\cup_i$ product on $C^*(K,L)$, $j : L \subset K$

we have a short exact sequence of cochain complex (with $\mathbb{Z}$-coefficient):

$$0 \to C^*(K,L) \xrightarrow{g^*} C^*(K) \xrightarrow{j^*} C^*(L) \to 0$$

$\forall\, u, v \in C^*(K,L)$

$$j^*(g^*u \cup_i g^*v) = j^*g^*u \cup_i j^*g^*v = 0$$

$\Rightarrow \exists !\ \alpha \in C^*(K,L)$.

s.t $g^*\alpha = g^*u \cup_i g^*v$.

Def:

$$u \cup_i v \triangleq \alpha \quad \text{i.e.} \quad g^*(u \cup_i v) = g^*u \cup_i g^*v.$$

② Coboundary formula for $\cup_i$-product on $C^*(K,L)$.

$$\delta(u \cup_i v) = (-1)^i \delta u \cup_i v + (-1)^{i+p} u \cup_i \delta v$$
$$- (-1)^i u \cup_{i-1} v - (-1)^{pq} v \cup_{i-1} u.$$

proof: Exercise. $\boxtimes$

① and ② tell us that we can obtain a homomorphism

$$Sq^i : H^p(K,L;\mathbb{Z}/2) \longrightarrow H^{p+i}(K,L;\mathbb{Z}/2)$$

by the same process as above (steps 1–4 in 1.2)

2. Some properties of Sq^i.

<u>2.1.</u> $Sq^i : H^p(K,L;\mathbb{Z}/2) \longrightarrow H^{p+i}(K,L;\mathbb{Z}/2)$ is a natural homomorphism.

proof: ① homomorphism. ✓ (by definition).

② natural ✓ ($\cup_i$ is functorial)

$$f:(K,L) \to (K',L')$$

$$\begin{array}{ccc} H^p(K',L';\mathbb{Z}/2) & \xrightarrow{Sq^i} & H^{p+i}(K',L';\mathbb{Z}/2) \\ \downarrow f^* & \circlearrowright & \downarrow f^* \\ H^p(K,L;\mathbb{Z}/2) & \xrightarrow{Sq^i} & H^{p+i}(K,L;\mathbb{Z}/2) \end{array}$$

<u>i.e.</u> $f^* Sq^i = Sq^i f^*$. ☒

<u>Rmk</u>: Sq^i independence the choice of φ.

<u>2.2</u> $Sq^i : H^p(K,L;\mathbb{Z}/2) \longrightarrow H^{p+i}(K,L;\mathbb{Z}/2)$ is zero if $i>p$ or $i<0$.

proof: by definition. ☒

<u>2.3.</u> $Sq^i u = u^2$ if $|u| = i$.

proof: $\cup_0$ is just the cup-product. ☒

<u>2.4</u> Sq^i commutes with the coboundary homomorphism

$$\delta^* : H^*(L) \to H^{*+1}(K,L)$$

Recall the definition of δ^*:

$\delta^*(\bar{a}) = \bar{c}$. where

$$\begin{array}{ccc} & & b \overset{j^*}{\longmapsto} a \\ & & \downarrow \delta \\ c & \overset{g^*}{\longmapsto} & \delta b \end{array}$$

<u>Proof of 2.4</u> suppose that $|a| = p \Rightarrow |c| = p+1$

$$Sq^i \bar{a} = Sq_{p-i}\bar{a} = \overline{a \cup_{p-i} a}.$$

$$Sq^i \delta^* \bar{a} = Sq^i \bar{c} = Sq_{p+1-i}\bar{c} = \overline{c \cup_{p+1-i} c}$$

since $q^*(c \cup_{p+1-i} c) = q^* c \cup_{p+1-i} q^* c$

$= \delta b \cup_{p+1-i} \delta b$

$= \delta(b \cup_{p+1-i} \delta b + b \cup_{p-i} b)$ (by Remark b))

and $j^*(b \cup_{p+1-i} \delta b + b \cup_{p-i} b) = j^* b \cup_{p+1-i} j^* \delta b + j^* b \cup_{p-i} j^* b$

$= a \cup_{p-i} a.$ ($j^* \delta b = \delta j^* b = \delta a = 0$)

So by definition of δ^*, we get that

$$\delta^* Sq^i \bar{a} = \overline{c \cup_{p+1-i} c} = Sq^i \delta^* \bar{a}.$$ $\square$

2.5 Sq^i commutes with suspension S^*

$$\begin{tikzcd}
\widetilde{H}^p(K;\mathbb{Z}/2) \arrow[r, "Sq^i"] \arrow[d, "S^*"] \arrow[dr, phantom, "\circlearrowright"] & \widetilde{H}^{p+i}(K;\mathbb{Z}/2) \arrow[d, "S^*"] \\
\widetilde{H}^{p+1}(SK;\mathbb{Z}/2) \arrow[r, "Sq^i"] & \widetilde{H}^{p+1+i}(SK;\mathbb{Z}/2)
\end{tikzcd}$$

Recall: cone over X: $CX = X \times I / X \times \{0\}$

suspension of X: $SX = X \times I / X \times \dot{I}$.

$$S^* : \widetilde{H}^*(X) \xrightarrow[\cong]{\delta^*} H^{p+1}(CX, X) \xrightarrow[\cong]{} \widetilde{H}^{p+1}(SX)$$ $\square$

3. An application.

Let $K = \mathbb{R}P^2$. $H^*(\mathbb{R}P^2) \cong \mathbb{Z}/2[\alpha]/\langle \alpha^3 \rangle$. $|\alpha| = 1$.

$Sq^1 \alpha = \alpha^2 \neq 0$. $\in H^2(K;\mathbb{Z}/2)$

so. $Sq^1 S^* \alpha = S^* Sq^1 \alpha \neq 0$. $\in H^3(SK;\mathbb{Z}/2)$

$\Rightarrow$ Sq^1 operat. on $H^2(SK;\mathbb{Z}/2)$ is non-trivial. $\square$